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1. Preliminaries.

We assume the reader to be familiar with the fundamental facts in set theory, topology and measure theory. In this chapter we will explain our terminology and notation that will be used throughout this book and discuss some facts that may not be emphasized in standard textbooks.

1.1. σ-algebras.

Let S be a set. Elements of S are denoted by $a, b, x, y, \ldots$. The class of all subsets of S is denoted by 2^S. Subclasses of 2^S are denoted by $\mathfrak{A}$, $\mathfrak{B}$, ... The (set-theoretical) union and the (set-theoretical) intersection are denoted by $\cup$ and $\cap$ respectively. It should be noted that

$$\begin{aligned} \mathfrak{A} \cup \mathfrak{B} \cup \mathfrak{C} &= \{A : A \in \mathfrak{A} \text{ or } A \in \mathfrak{B} \text{ or } A \in \mathfrak{C}\} \\ &\neq \{A \cup B \cup C : A \in \mathfrak{A}, B \in \mathfrak{B}, C \in \mathfrak{C}\} . \end{aligned}$$

The (set-theoretical) differece is denoted by $\setminus$:

$$A \setminus B = \{x : x \in A, x \notin B\} .$$

The disjoint union is denoted by $\sum$ or $+$ and the proper difference by $-$:

$$A = \sum_{i \in I} A_i \iff A_i \cap A_j = \emptyset \ (i \neq j) \text{ and } A = \bigcup_{i \in I} A_i ,$$

$$C = A + B \iff A \cap B = \emptyset \text{ and } C = A \cup B ,$$

$$C = A - B \iff A \supset B \text{ and } C = A \setminus B .$$

When we use the notation $\{A_n\}_n$ or A_n, $n = 1,2, \ldots,$ it means a finite or infinite sequence. If we use the phrase "a sequence $\{A_n\}_n$ " or "a sequence $A_n, n = 1,2, \ldots$", then it always means an infinite sequence.

Now we will introduce several special classes of subsets of S. Let $\mathcal{A}$ be a class of subets of S.

$\mathcal{A}$ is called a <u>comlementary class</u> on S, if it is closed under complements, i.e. $A \in \mathcal{A} \Longrightarrow A^c \in \mathcal{A}$.

$\mathcal{A}$ is called a <u>multipicative class</u> on S, if it is closed under finite intersections.

$\mathcal{A}$ is called a <u>Dynkin class</u> on S, if it contains S and is closed under countable disjoint unions and proper differences.

$\mathcal{A}$ is called an <u>algebra</u> (resp. <u>σ-algebra</u>) on S, if it is non-empty and is closed under complements and finite (resp. countable) unions.

For a class $\mathcal{C} \subset 2^S$, the intersection of all σ-algebras on S including $\mathcal{C}$ is a σ-algebra. It is called the σ-algebra <u>generated</u> by $\mathcal{C}$, $\sigma[\mathcal{C}]$ in notation. The same terminology can be used for other classes.

Let $\{\mathcal{B}_i\}_{i\in I}$ be a family of σ-algebras on S. the intersection $\bigcap_{i\in I} \mathcal{B}_i$ is the largest σ-algebra on S included in every $\mathcal{B}_i$, $i\in I$. But the union $\bigcup_{i\in I} \mathcal{B}_i$, i.e. $\sigma[\bigcup_{i\in I} \mathcal{B}_i]$, is the smallest σ-algebra on S including every $\mathcal{B}_i$, $i\in I$. It is called the <u>lattice union</u> of $\{\mathcal{B}_i\}_{i\in I}$ and is denoted by $\bigvee_{i\in I} \mathcal{B}_i$.

Σ $\mathcal{B}$

From the definition we can easily derive the following :

(σ. 1) $\quad \mathcal{C}_1 \subset \mathcal{C}_2 \quad \Longrightarrow \quad \sigma[\mathcal{C}_1] \subset \sigma[\mathcal{C}_2]$

(σ. 2) $\quad \sigma[\sigma[\mathcal{C}]] = \sigma[\mathcal{C}]$

(σ. 3) $\quad \sigma[\bigcup_{i\in I} \mathcal{C}_i] = \bigvee_{i\in I} \sigma[\mathcal{C}_i]$.

<u>Theorem 1.1.1</u> (<u>The Dynkin class theorem</u>). Every Dynkin class including a multiplicative class $\mathcal{C}$ includes $\sigma[\mathcal{C}]$.

<u>Proof</u>. Let $\delta[\mathcal{C}]$ dnote the smallest Dynkin class including $\mathcal{C}$,

It is enough to prove that

$$\delta[\mathcal{C}] \supset \sigma[\mathcal{C}].$$

To do this, it is enough to show that $\delta[\mathcal{C}]$ is a σ-algebra.

Since $\delta[\mathcal{C}]$ is a Dynkin class, it suffices to show that

(1) $\quad A, B \in \delta[\mathcal{C}] \quad \Longrightarrow \quad A \cap B \in \delta[\mathcal{C}]$.

Let $\mathcal{D}_1$ denote the class

$$\{B \subset S \ : \ A \cap B \in \delta[\mathcal{C}] \quad \text{for any} \ A \in \mathcal{C}\}.$$

Since $\mathcal{C}$ is multiplicative ,it is easy to check that $\mathcal{D}_1$ is a Dynkin class including $\mathcal{C}$, showing that $\mathcal{D}_1 \supset \delta[\mathcal{C}]$.

Therefore

(2) $\quad A \in \mathcal{C} \ , \ B \in \delta[\mathcal{C}] \quad \Longrightarrow \quad A \cap B \in \delta[\mathcal{C}]$.

Consider the class

$$\mathcal{D}_2 = \{\cancel{A}\,B \subset S \ : \ A \cap B \in \delta[\mathcal{C}] \quad \text{for any} \ A \in \delta[\mathcal{C}]\}$$

Using (2), we can easily check that $\mathcal{D}_2$ is a Dynkin class including $\mathcal{C}$, showing that $\mathcal{D}_2 \supset \delta[\mathcal{C}]$. This implies (1).

<u>Theorem 1.1.2</u>. If $\mathcal{C}$ is a non-empty complementary class on S, then $\sigma[\mathcal{C}]$ is the smallest class $\mathcal{B}_0$ including $\mathcal{C}$ and closed under countable disjoint unions and countable intersections.

<u>Proof</u>. Let $\mathcal{B}_1 = \{B \subset S : B, B^c \in \mathcal{B}_o\}$. Since $\mathcal{C}$ is complementary, $\mathcal{B}_1 \supset \mathcal{C}$. $\mathcal{B}_1$ is obviously complementary. Let $B_n \in \mathcal{B}_1$, $n = 1, 2, \ldots$ Then B_n, $B_n^c \in \mathcal{B}_o$. Therefore we have

$$\bigcup_n B_n = \sum_n B_1^c \cap B_2^c \cap \ldots \cap B_{n-1}^c \cap B_n \in \mathcal{B}_o$$

and $\left(\bigcup_n B_n\right)^c = \bigcap_n B_n^c \in \mathcal{B}_o$

implying that $\bigcup_n B_n \in \mathcal{B}_1$. Therefore $\mathcal{B}_1$ is a σ-algebra including $\mathcal{C}$. This implies that

$$\mathcal{B}_o \supset \mathcal{B}_1 \supset \sigma[\mathcal{C}]$$

It is obvious that $\mathcal{B}_o \subset \sigma[\mathcal{C}]$.

Let f be a map from S into T and $\mathcal{C}$ be a class of subsets of T. The class $\{f^{-1}(C) : C \in \mathcal{C}\}$ is denoted by $f^{-1}(\mathcal{C})$.

<u>Theorem 1.1.3.</u> $f^{-1}(\sigma[\mathcal{C}]) = \sigma[f^{-1}(\mathcal{C})]$.

<u>Proof.</u> Observing that

$$f^{-1}(B^c) = f^{-1}(B)^c \quad \text{and} \quad f^{-1}\left(\bigcup_n B_n\right) = \bigcup_n f^{-1}(B_n),$$

we can easily check that $f^{-1}(\sigma[\mathcal{C}])$ is a σ-algebra on S including $f^{-1}(\mathcal{C})$. Therefore

$$f^{-1}(\sigma[\mathcal{C}]) \supset \sigma[f^{-1}(\mathcal{C})].$$

Similarly we can prove that the class

$$\mathcal{B} = \{B \subset T : f^{-1}(B) \in \sigma[f^{-1}(\mathcal{C})]\}$$

is a σ-algebra on T including $\mathcal{C}$. Hence we have $\sigma[\mathcal{C}] \subset \mathcal{B}$, which implies that

$$f^{-1}(\sigma[\mathcal{C}]) \subset \sigma[f^{-1}(\mathcal{C})].$$

For $\mathcal{A} \subset 2^S$ and $T \subset S$, the class

$$\mathcal{A} \cap T = \{ A \cap T \ : \ A \in \mathcal{A} \}$$

is called the <u>trace</u> of $\mathcal{A}$ on T. If $\mathcal{A}$ is a σ-algebra on S, then $\mathcal{A} \cap T$ is a σ-algebra on T, called the <u>trace σ-algebra</u> of $\mathcal{A}$ on ~~S~~. If the map $i : T \to S$ is given by $x \mapsto x$ (<u>canonical injection</u>), then

$$i^{-1}(\mathcal{A}) = \mathcal{A} \cap T \quad \text{for every} \quad \mathcal{A} \subset 2^S.$$

Let $\{\mathcal{B}_i\}_{i \in I}$ be a family of σ-algebras on S, where I is an arbitrary index set. The intersection $\bigcap_{i\in I} \mathcal{B}_i$ is also a σ-algebra. It is the largest σ-algebra on S included in every $\mathcal{B}_i$, $i \in I$, but the union $\mathcal{C} = \bigcup_{i \in I} \mathcal{B}_i$ is not a σ-algebra on S in general. The σ-algebra generated by $\mathcal{C}$ is the smallest σ-algebra including every $\mathcal{B}_i$, $i \in I$. It is called the <u>lattice union</u> of $\mathcal{B}_i$, $i \in I$, and is denoted by $\bigvee_{i \in I} \mathcal{B}_i$:

$$\bigvee_{i \in I} \mathcal{B}_i = \sigma\Big[\bigcup_{i \in I} \mathcal{B}_i\Big] .$$

<u>Theorem 1.1.4.</u> $f^{-1}(\bigvee_{i \in I} \mathcal{B}_i) = \bigvee_{i \in I} f^{-1}(\mathcal{B}_i)$.

<u>Proof.</u> Use Theorem 1.1.3 to obtain

$$f^{-1}(\sigma[\textstyle\bigcup_i \mathcal{B}_i]) = \sigma[f^{-1}(\bigcup_i \mathcal{B}_i)] = \sigma[\bigcup_i f^{-1}(\mathcal{B}_i)].$$

Let $S = \prod_{i \in I} S_i$ (<u>Cartesian product</u>) and for each $i \in I$, let $\mathcal{B}_i$ be a σ-algebra on S_i. The canonical projection from S into S_i given by

$$x \mapsto \text{the } i\text{-th component of } x$$

is denoted by p_i. It is obvious that $p_i^{-1}(\mathcal{B}_i)$ is a σ-algebra on S. The lattice union $\bigvee_{i \in I} p_i^{-1}(\mathcal{B}_i)$ is called the <u>product</u>

<u>σ-algebra</u> of $\mathcal{B}_i$, $i \in I$, and is denoted by $\prod_{i \in I} \mathcal{B}_i$. Note that this is not the Cartesian product in the set-theoretical sense, even though we use the same symbol $\prod$. A subset A of $S = \prod_i S_i$ is called <u>σ-determined</u> if we have a countable subset $J = J(A)$ of I such that

$$p_i(x) = p_i(y) \text{ for every } i \in J \quad \Rightarrow \quad 1_A(x) = 1_A(y),$$

where 1_A is the <u>indicator</u> of A, i.e.

$$1_A(x) = 1 \text{ for } x \in A, = 0 \text{ for } x \in A^c$$

<u>Theorem 1.1.5</u>. Every set $B \in \prod_{i \in I} \mathcal{B}_i$ is σ-determined.

<u>Proof.</u> The class $\mathcal{B}$ of all σ-determined subset of $S = \prod_i S_i$ is a σ-algbra on S. For each i, every set A in $p_i^{-1}(\mathcal{B}_i)$ is σ-determined, take the singleton $\{i\}$ for J(A). Therefore $p_i^{-1}(\mathcal{B}_i) \subset \mathcal{B}$ for every $i \in I$. Therefore we have

$$\mathcal{B} \supset \bigvee_i p_i^{-1}(\mathcal{B}_i) = \prod_i \mathcal{B}_i ,$$

which completes the proof.

A σ-algebra on S is called <u>σ-generated</u> if it is generated by a countable class of subsets of S.

<u>Theorem 1.1.6</u>. Let $\mathcal{B}_n$ be a σ-generated σ-algebra on S_n for each $n = 1,2,\ldots$. Then the product σ-algebra $\mathcal{B} = \prod_n \mathcal{B}_n$ on $S = \prod_n S_n$ is σ-generated.

<u>Proof.</u> Let $\mathcal{C}_n$ be a countable class generating $\mathcal{B}_n$. Then the union $\mathcal{C} = \bigcup_n p_n^{-1}(\mathcal{C}_n)$ is countable. Since

$$\mathcal{C} \subset \bigcup_n p_n^{-1}(\mathcal{B}_n) \subset \mathcal{B},$$

we have $\sigma[\mathcal{C}] \subset \mathcal{B}$. Using Theorem 1.1.3, we obtain

$$\sigma[\mathcal{C}] \supset \sigma[p_n^{-1}(\mathcal{C}_n)] = p_n^{-1}(\sigma[\mathcal{C}_n]) = p_n^{-1}(\mathcal{B}_n)$$

for every n. Therefore $\sigma[\mathcal{C}] \supset \mathcal{B}$.

A class $\mathcal{O}$ of subsets of S is said to separate x and y if there exists a set A in $\mathcal{O}$ such that $1_A(x) \neq 1_A(y)$. $\mathcal{O}$ is called a <u>separating class</u> on S if $\mathcal{O}$ separates every two distinct points in S.

Theorem 1.1.7. $\mathcal{O}$ is a separating class on S if and only if $\sigma[\mathcal{O}]$ is separating.

Proof. If $\mathcal{O}$ is separeting, then $\sigma[\mathcal{O}]$ is obvionsly separating. Suppose that $\mathcal{O}$ is not separating. Then we have two distinct points $x, y \in S$ such that

$$1_A(x) = 1_A(y) \quad \text{for every } A \in \mathcal{O}.$$

Let B denote the class of all A such that $1_A(x) = 1_A(y)$. Then B is a σ-algbra on S including $\mathcal{O}$, because

$$1_B = 1 - 1_A \quad \text{for } B = A^c$$

and

$$1_C = \sup_n 1_{A_n} \quad \text{for } C = \bigcup_n A_n.$$

Thus σ is not sep

1.2. Measurability of maps.

Let $\mathcal{B}_i$ be a σ-algebra on S_i for $i = 1, 2$. A map $f : S_1 \to S_2$ is called measurable $\mathcal{B}_1/\mathcal{B}_2$, $f \in \mathcal{B}_1/\mathcal{B}_2$ in notation, if

$$f^{-1}(\mathcal{B}_2) \subset \mathcal{B}_1 \text{, i. e. } f^{-1}(B_2) \in \mathcal{B}_1 \quad \text{for } B_2 \in \mathcal{B}_2.$$

Theorem 1.2.1. If $f^{-1}(\mathcal{C}_2) \subset \mathcal{B}_1$ for some class $\mathcal{C}_2$ generating $\mathcal{B}_2$, then f is measurable $\mathcal{B}_1/\mathcal{B}_2$.

Proof. $f^{-1}(\sigma[\mathcal{C}_2]) = \sigma[f^{-1}(\mathcal{C}_2)]$ by Theorem 1.1.3. Therefore

$$f^{-1}(\mathcal{B}_2) \subset \sigma[\mathcal{B}_1] \quad \text{by the assumption.}$$

Let $\mathcal{B}$ be a σ-algebra on S and let $T \subset S$. Then the canonical injection $i = i_{T,S} : T \to S$ is measurable $\mathcal{B} \cap T/\mathcal{B}$.

Let f be a map from S_1 into S_2, let $T_i \subset S_i$ for $i = 1, 2$, and suppose that $f(T_1) \subset T_2$. The map

$$\begin{aligned} g \ : \ & T_1 \to T_2 \\ & x \mapsto f(x) \end{aligned}$$

is called the restriction of f to (T_1, T_2), $f|_{T_1,T_2}$ in notation. It is denoted by $f|_{T_1}$ if $T_2 = S_2$.

If $f \in \mathcal{B}_1/\mathcal{B}_2$, then $f|_{T_1,T_2} \in \mathcal{B}_1 \cap T_1/\mathcal{B}_2 \cap T_2$, because

$$\begin{aligned} (f|_{T_1,T_2})^{-1}(B_2 \cap T_2) &= f^{-1}(B_2 \cap T_2) \cap T_1 \\ &= f^{-1}(B_2) \cap f^{-1}(T_2) \cap T_1 \\ &= f^{-1}(B_2) \cap T_1 \quad \text{by } f(T_1) \subset T_2 \\ &\in \mathcal{B}_1 \cap T_1 \quad \text{for } B_2 \in \mathcal{B}_2. \end{aligned}$$

Let f be a map from S_1 into S_2 and g be a map from S_2 into S_3. Then the map

$$h : S_1 \to S_3$$
$$x \mapsto g(f(x))$$

is called the composite map of f and g, $g \circ f$ in notion. If $f \in \mathcal{B}_1 / \mathcal{B}_2$ and $g \in \mathcal{B}_2 / \mathcal{B}_3$, then $g \circ f \in \mathcal{B}_1 / \mathcal{B}_3$, because

$$(g \circ f)^{-1}(\mathcal{B}_3) = f^{-1}(g^{-1}(\mathcal{B}_3)) \subset f^{-1}(\mathcal{B}_2) \subset \mathcal{B}_1 .$$

Let f_i be a map from S into T_i for $i \in I$. The map

$$f : S \to \prod_{i \in I} T_i$$
$$x \longmapsto (f_i(x))_{i \in I}$$

is called the <u>product map</u> of f_i, $i \in I$, denoted by $\prod_{i \in I} f_i$. It is obvious that

$$f_i = p_i \circ f \quad \text{for} \quad i \in I.$$

If $f_i \in \mathcal{B} / \mathcal{C}_i$ for every $i \in I$, then

$$f = \prod_i f_i \in \mathcal{B} / \prod_i \mathcal{C}_i ,$$

because we can use Theorem 1.1.4 to obtain

$$f^{-1}(\prod_i \mathcal{C}_i) = f^{-1}(\bigvee_i p_i^{-1}(\mathcal{C}_i)) = \bigvee_i f^{-1}(p_i^{-1}(\mathcal{C}_i))$$
$$= \bigvee_i f_i^{-1}(\mathcal{C}_i) \subset \mathcal{B} .$$

Let f_i be a map from S_i into T_i for $i \in I$. The map

$$f \ : \ \prod_{i \in I} S_i \longrightarrow \prod_{i \in I} T_i$$

$$(x_i)_{i \in I} \longmapsto (f_i(x_i))_{i \in I}$$

is called the bilateral product map of f_i, $i \in I$, denoted by $\prod^b_{i \in I} f_i$.

It is easy to see that

$$\prod^b_{i \in I} f_i = \prod_{i \in I} f_i \circ p_i \ .$$

Using the results proved above, we can prove that if $f_i \in \mathcal{B}_i / \mathcal{C}_i$ for every $i \in I$, then $\prod^b_i f_i \in \prod_i \mathcal{B}_i / \prod_i \mathcal{C}_i$.

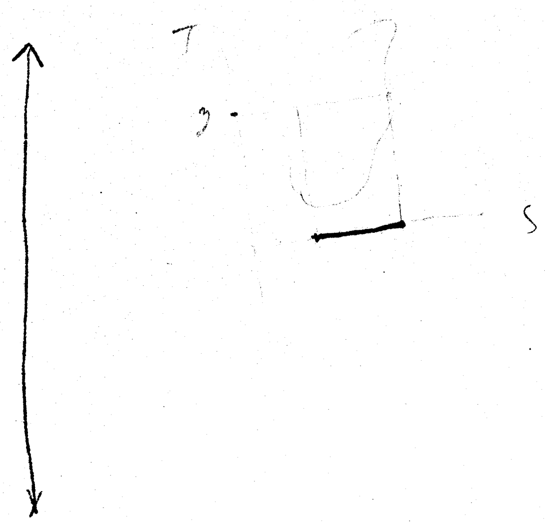
Thus we obtain the following :

<u>Theorem 1.2.2.</u> Measurability is inherited by composite maps, restrictions, product maps and bilateral product maps.

Let A be a subset of $S \times T$. For any $y \in T$ the set

$$\{ x \in S \ : \ (x,y) \in A \}$$

is called the <u>section</u> (or section set) of A at $y \in T$, $A(y)$ in notation. Similarly for the section $A(x)$ of A at $x \in S$.



Let f be a map from $S \times T$ into U. For any $y \in T$ the map

$$f_y : S \to U$$
$$x \mapsto f(x,y)$$

is called the <u>section</u> (or <u>section map</u>) of f for $y \in T$. Similarly for the section f_x of f for $x \in S$. The section map of the indicator 1_A of A for $y \in T$ (or $x \in S$) is the indicator of the section set of A for $y \in T$ (or $x \in S$).

<u>Theorem 1.2.3.</u> Let $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{U}$ be σ-algebras on S, T and U respectively.

(i) $A \in \mathcal{S} \times \mathcal{T} \Rightarrow A(y) \in \mathcal{S}$ and $A(x) \in \mathcal{T}$.

(ii) $f \in \mathcal{S} \times \mathcal{T}/\mathcal{U}$ $f_y \in \mathcal{S}/\mathcal{U}$ and $f_x \in \mathcal{T}/\mathcal{U}$.

<u>Proof.</u>

(i) Let $\mathcal{B}$ denote the class of all $A \subset S \times T$ such that $A(y) \in \mathcal{S}$ for every $y \in T$. It is easy to check that $\mathcal{B}$ is a σ-algebra on $S \times T$. If $A = E \times F$, $E \in \mathcal{S}$, $F \in \mathcal{T}$, then

$$A(y) = \begin{cases} E & \text{for } y \in F \\ \emptyset & \text{for } y \notin F, \end{cases}$$

so $A(y) \in \mathcal{B}$. Therefore $\mathcal{B} \supset \mathcal{S} \times \mathcal{T}$, proving that $A(y) \in \mathcal{S}$ for $A \in \mathcal{S} \times \mathcal{T}$. Similarly $A(x) \in \mathcal{T}$ for $A \in \mathcal{S} \times \mathcal{T}$.

(ii) Immediate from (i).

1.3. Borel spaces.

A set S endowed with a σ-algebra $\mathcal{S}$ on S is called a Borel space, which is denoted by $S(\mathcal{S})$ or $(S,\mathcal{S})$. S and $\mathcal{S}$ are called the base set and the Borel structure of $S(\mathcal{S})$ respectively. When we call a set S a Borel space, we agree that a certain Borel structure is assigned to S. A subset B of $S(\mathcal{S})$ is called a Borel set if $B \in \mathcal{S}$. Unless otherwise stated, a subset T of a Borel space $S(\mathcal{S})$ will be regarded as a Borel space with the trace σ-algebra $\mathcal{S} \cap T$, called a Borel subspace, and the Cartesian product of Borel spaces $S_i(\mathcal{S}_i)$, $i \in I$, will be regarded as a Borel space with the product σ-algebra $\prod_{i\in I} \mathcal{S}_i$, called the Borel product.

A map f from a Borel space $S(\mathcal{S})$ into another Borel space $T(\mathcal{T})$ is called Borel measurable if it is measurable $\mathcal{S}/\mathcal{T}$, i.e. $f^{-1}(\mathcal{T}) \subset \mathcal{S}$. Borel measurability is inherited by composite maps, restrictions, product maps and bilateral product maps by Theorem 1.2.2. A map $f: S(\mathcal{S}) \to T(\mathcal{T})$ is called Borel bimeasurable if f is bijective and $f(\mathcal{S}) = \mathcal{T}$. In this case both f and f^{-1} are Borel measurable. If there exists a Borel bimeasurable map $f: S(\mathcal{S}) \to T(\mathcal{T})$, $T(\mathcal{T})$ is called Borel isomorphic to $S(\mathcal{S})$, $T(\mathcal{T}) \sim S(\mathcal{S})$ in notation. If we want to refer to a map $f: S(\mathcal{S}) \longrightarrow T(\mathcal{T})$ showing $T(\mathcal{T}) \sim S(\mathcal{S})$, we say that $T(\mathcal{T})$ is Borel isomorphic to $S(\mathcal{S})$ under f, " $T(\mathcal{T}) \sim S(\mathcal{S})$ (f) " in notation. Borel isomorphism is an equivalence relation. Two Borel spaces are said to have the same Borel type if they are Borel isomorphic to each other.

In the discussion below S, T, S_n, T_n, ...stand for Borel spaces.

Theorem 1.3.1.

(i) $T \sim S\ (f)$, $S_1 \subset S$, $T_1 = f(S_1) \Longrightarrow T_1 \sim S_1\ (f|_{S_1, T_1})$,

(ii) $T_i \sim S_i\ (f_i)$, $i \in I$, $\Longrightarrow \prod_i T_i \sim \prod_i S_i\ (\prod_i^b f_i)$.

Proof. Obvious by Theorem 1.2.2.

Theorem 1.3.2. If

$$S = \sum_n S_n, \quad S_n : \text{Borel in } S$$

and $T = \sum_n T_n$, T_n : Borel in S,

and if $T_n \sim S_n$ for $n = 1.2, \ldots,$ then

$$T \sim S.$$

proof. Let $f_n : S_n \longrightarrow T_n$ be Borel bimeasurable. Then the map

$$f : S \to T$$
$$x \mapsto f_n(x) \quad (x \in S_n), \quad n = 1, 2, \ldots,$$

is Borel bimeasurable.

The following theorem corresponds to Bernstein's theorem in set theory.

Theorem 1.3.3. If

$$S \sim T_1 \subset T, \quad T_1 : \text{Borel in } T$$

and
$$T \sim S_1 \subset S, \quad S_1 : \text{Borel in } S,$$

then
$$S \sim T.$$

Proof. Let $f : S \to T_1$ and $g : T \longrightarrow S_1$ be borel bimeasurable. We define S_n and T_n for $n = 1, 2, \ldots$ as follows :

$$S_2 = g(T_1), \quad T_2 = f(S_1), \ldots, \quad S_n = g(T_{n-1}), \quad T_n = f(S_{n-1}), \ldots.$$

Then

$$S \supset S_1 \supset S_2 \supset \ldots \quad \text{and} \quad T \supset T_1 \supset T_2 \supset \ldots.$$

Denote $\bigcap_n S_n$ and and $\bigcap_n T_n$ by S_∞ and T_∞ respectively. Then

$$S = (S-S_1) + (S_1-S_2) + (S_2-S_3) + (S_3-S_4) + \dots + S_\infty$$

$$T = (T-T_1) + (T_1-T_2) + (T_2-T_3) + (T_3-T_4) + \dots + T_\infty,$$

where the sets connected by lines are Borel isomorphic to each other under appropriate restrictions of f and g. Since all these sets are Borel in S or in T, we can apply Theorem 1.3.2 to conclude that $S \sim T$.

Remark. Let S be a set, $\mathcal{B}$ a σ-algebra on S and $T = T(\mathcal{T})$ a Borel space. If a map $f: S \to T$ is measurable $\mathcal{B}/\mathcal{T}$, f is called *measurable $\mathcal{B}$* or *measurable with respect to $\mathcal{B}$*. Let $\mathcal{B}_i$ be a σ-algebra ? on S_i for $i \in I$. Then the product σ-algebra $\prod_i \mathcal{B}_i$ on the product space $S = \prod_i S_i$ is the smallest σ-algebra on S with respect to which all canoniacal projections $p_i : S \to S_i$, $i \in I$, are measurable.

1.4. Topological spaces.

Let S be a topological space. The class of all open subsets of S is called the open system (or topological structure or simply topology) on S, $\mathcal{O}(S)$ in notation. The σ-algebra generated by the open subsets of S, i.e. $\sigma[\mathcal{O}(S)]$, is called the topological σ-algebra on S, $\mathcal{B}(S)$ in notation.

A subclass $\mathcal{U}$ of $\mathcal{O}(S)$ is called an open base in S if every open set can be expressed as a union (finite or not) of sets in $\mathcal{U}$. A subclass $\mathcal{V}$ of $\mathcal{O}(S)$ is called an open subbase in S, if the finite intersections of sets in $\mathcal{V}$ form an open base in S, i.e. if the multiplicative class generated by $\mathcal{V}$ is an open base in S. It is obvious that every open base is an open subbase.

An open set containing x is called a neighborhood of x and is denoted by $U(x)$, $V(x)$ or $W(x)$. A class $\mathcal{U}$ of neighborhoods of x is called a neighborhood base of x if every neighborhood of x includes at least one neighborhood belonging to $\mathcal{U}$.

Let $\{A_n\}$ be a sequence of subsets of S and a be a point in S. If the following three conditions are satisfied, then we say that $\{A_n\}$ monotonically converges to a, $A_n \downarrow a$ in notation :

(i) $A_n \ni a$, $n = 1,2,\ldots,$

(ii) $A_1 \supset A_2 \supset A_3 \supset \ldots,$

(iii) for every neighborhood $U(x)$ there exists at least one set $A_m \subset U(x)$ (then $A_n \subset U(x)$ for every $n > m$).

If $f : S \to T$ is continuous at $a \in S$, then

$$A_n \downarrow a \Rightarrow f(A_n) \downarrow f(a).$$

If S is Hausdorff, then

$$A_n \downarrow a \implies \bigcap_n A_n = \bigcap_n \bar{A}_n = a ,$$

because for $b \neq a$, we can find a neighborhood $U(a)$ with $U(a) \not\ni b$ by the Hausdorff property of S.

Every subset T of a topological space S will be regarded as a topological space with the <u>relative topology</u> $\mathcal{O}(T) = \mathcal{O}(S) \cap T$. In this case T is called a (<u>topological</u>) <u>subspace</u> of S. The Cartestian product $S = \prod_{i \in I} S_i$ of a family of topological spaces S_i, $i \in I$, will be regarded as a topological space with the product topology.

In this case S is called the (<u>topological</u>) <u>product</u> of S_i, $i \in I$. The class

$$\mathcal{V} = \bigcup_{i \in I} p_i^{-1}(\mathcal{O}(S_i))$$

is an open subbase of the topological product $\prod_{i \in I} S_i$.

T is said to be <u>homeomorphic</u> to S, $T \approx S$ in notation, if there exists a bicontinuous map $f : S \to T$ (= a bijection $f : S \to T$ such that both f and f^{-1} are continuous).

T is said to be <u>1-1 dominated</u> by S, $T \underset{1-1}{\prec} S$ in notation, if there exists a continuous bijection $f : S \to T$.

T is said to be <u>dominated</u> by S, $T \prec S$ in notation, if there exists a continuous surjection $f : S \to T$.

If we want to refer to a map $f : S \to T$ for which $T \approx S$, we say that T is homeomorphic to S under f, $T \approx S$ (f) in notation, Similarly for other relations.

It is easy to check the following :

1. $\approx$ is an equivalence relation,
2. $T \approx S \Rightarrow T \underset{1-1}{\prec} S \Rightarrow T \prec S$,
3. $T \prec S$, $S \prec U \Rightarrow T \prec U$,
4. $T \underset{1-1}{\prec} S$, $S \underset{1-1}{\prec} U \Rightarrow T \underset{1-1}{\prec} U$,
5. $T_i \prec S_i$ $(i \in I) \Rightarrow \prod_i T_i \prec \prod_i S_i$, and similarly for $\underset{1-1}{\prec}$ and $\approx$.

6. $T \prec S \ (f), T' \subset T, S' = f^{-1}(T') \implies T' \prec S'$,
and similarly for $\underset{1-1}{\prec}$ and $\approx$.

If every open covering of S has a finite (resp. countable) subcovering, then S is called <u>compact</u> (resp. <u>Lindelöf</u>). If every subspace of S is Lindelöf, then S is said to be <u>fully Lindelöf</u>. S is fully Lindelof if and only if every class of open subsets of S has a countable subclass with the same union. If S is fully Lindelöf, then for every open base $\mathcal{U}$ in S , every open subset of S is expressible as a countable union of sets in $\mathcal{U}$.

<u>Theorem 1.4.1</u>. $\mathcal{B}(S)$ is the smallest class including all open sets and all closed sets and closed under countable disjoint unions and countable intersections.

<u>Proof</u>. Obvious by Theorem 1.1.2.

<u>Theorem 1.4.2.</u>
(i) $\sigma[\mathcal{V}] \subset \mathcal{B}(S)$ for every open subbase $\mathcal{V}$ in S .
(ii) If S is fully Lindelöf, then the above two σ-algebras are the same.

<u>Proof</u>. (i) is obvious by $\mathcal{V} \subset \mathcal{O}(S)$. To prove (ii), denote by $\mathcal{U}$ the multiplicative class generated by $\mathcal{V}$. Then $\mathcal{U}$ is an open base and $\mathcal{U} \subset \sigma[\mathcal{V}]$. Since S is fully Lindelöf, every open set is expressible as a countable union of sets in $\mathcal{U}$. Hence we have $\mathcal{O}(S) \subset \sigma[\mathcal{U}]$. Therefore we have

$$\mathcal{B}(S) = \sigma[\mathcal{O}(S)] \subset \sigma[\mathcal{U}] \subset \sigma[\mathcal{V}],$$

implying $\mathcal{B}(S) = \sigma[\mathcal{V}]$ by (i).

Theorem 1.4.3. If T is a subspace of S, then

$$\mathcal{B}(T) = \mathcal{B}(S) \cap T .$$

Proof. For the canonical injection $i : T \to S$ we have $i^{-1}(\mathcal{C}) = \mathcal{C} \cap T$ for every $\mathcal{C} \subset 2^S$. Therefore

$$\begin{aligned}\mathcal{B}(T) &= \sigma[\mathcal{O}(T)] = \sigma[\mathcal{O}(S) \cap T] = \sigma[i^{-1}(\mathcal{O}(S))] \\ &= i^{-1}(\sigma[\mathcal{O}(S)]) \qquad \text{by Theorem 1.1.3.} \\ &= i^{-1}(\mathcal{B}(S)) = \mathcal{B}(S) \cap T .\end{aligned}$$

Theorem 1.4.4.

(i) $(\prod_{i \in I} S_i) \supset \prod_{i \in I} \mathcal{B}(S_i)$

(ii) If $\prod_{i \in I} S_i$ is fully Lindelöf, then these two σ-algebras are the same.

Proof. First observe that

$$\begin{aligned}\prod_i \mathcal{B}(S_i) &= \bigvee_i p_i^{-1}(\mathcal{B}(S_i)) = \bigvee_i p_i^{-1}(\sigma[\mathcal{O}(S_i)]) \\ &= \bigvee_i \sigma[p_i^{-1}(\mathcal{O}(S_i))] \qquad \text{by Theorem 1.1.3.} \\ &= \sigma[\bigcup_i p_i^{-1}(\mathcal{O}(S_i))] .\end{aligned}$$

Since $\mathcal{V} = \bigvee_i p_i^{-1}(\mathcal{O}(S_i))$ is an open subbase in the topological product $\prod_i S_i$, our theorem follows immediately from Theorem 1.4.2.

Since every countable product of topological spaces, each having a countable open base, also has a countable open base, it is fully Lindelöf. Therefore we can use Theorem 1.4.4. (ii) to obtain the following:

Theorem 1.4.5. If I is countable and if S_i has a countable open base for every $i \in I$, then

$$\mathcal{B}(\prod_{i \in I} S_i) = \prod_{i \in I} \mathcal{B}(S_i) .$$

Let us give an example of $\mathcal{B}(\prod_i S_i) \neq \prod_i \mathcal{B}(S_i)$. Let $S_i = \underline{\underline{R}}$ (the space of all real numbers with the usual topology) for every $i \in I$, where I is not countable. Then every singleton of $S = \prod_i S_i$ belongs to $\mathcal{B}(S)$, being a closed subset of S, but no singleton belongs to $\prod_i \mathcal{B}(S_i)$ by Theorem 1.1.5.

A topological space S will be regarded as a Borel space with the topological σ-algebra $\mathcal{B}(S)$, unless otherwise stated. Therefore we can define Borel sets, Borel measurable maps and Borel isomorphism for toplogical spaces. A countable intersection of open subsets of S is called a G_δ set and a countable union of closed subsets of S is an F_σ set. All G_δ sets and all F_σ sets are Borel sets. If $f : S \to T$ is continuous, then $f^{-1}(\mathcal{O}(T)) \subset \mathcal{O}(S)$. Using $\sigma(f^{-1}(\mathcal{C})) = f^{-1}(\sigma[\mathcal{C}])$ (Theorem 1.1.3.), we can easily prove the following.

Theorem 1.4.6.

(i) Every continuous map is Borel measurable.

(ii) Homeomorphism implies Borel isomorphism.

Let T be a topological subspace of S . Then the Borel space $T(\mathcal{B}(T))$ is a Borel subspace of the Borel space $S(\mathcal{B}(S))$, because $\mathcal{B}(T) = \mathcal{B}(S) \cap T$ (Theorem 1.4.3.).

Let S be the topological product of S_i , $i \in I$. Then the Borel space $S(\mathcal{B}(S))$ is the Borel product of Borel spaces $S_i(\mathcal{B}(S_i))$, $i \in I$, provided $\mathcal{B}(\prod_i S_i) = \prod_i \mathcal{B}(S_i)$. By Theorem 1.4.4. (ii) (or Theorem 1.4.5.) this condition is satisfied in most cases useful to probability theory ; see Theorem 2.5.8.

We will list some special topological spaces which will appear in this book frequently.

(a) $\underline{\underline{R}}$ = the real numbers with the usual topology (the <u>real line</u>).

$\underline{\underline{R}}^n$ = the topological product of n copies of $\underline{\underline{R}}$ $(n=1,2,\ldots)$.

$\underline{\underline{R}}^\infty$ = the topological product of a countably infinite number of copies of $\underline{\underline{R}}$.

$\underline{\underline{R}}^n$ has a countable open base for $n=1,2,\ldots,\infty$.

R^n is called the <u>real n-space</u> for $n=1,2,\ldots$ and $\underline{\underline{R}}^\infty$ is called the <u>real sequence space</u>. The topological σ-algebra $\mathcal{B}(\underline{\underline{R}}^n)$ is denoted by $\mathcal{B}^n$ for $n=1,2,\ldots,\infty$.

(b) The following subsets of $\underline{\underline{R}}$ are topological spaces with the relative topology:

$\underline{\underline{Q}}$ = the rational numbers,

$\underline{\underline{J}}$ = the irrational numbers,

$\underline{\underline{I}}$ = the unit interval $[0, 1] = \{x : 0 \leq x \leq 1\}$,

$\underline{\underline{N}}$ = the natural numbers,

$\underline{\underline{Z}}$ = the integers,

$\underline{\underline{2}} = \{0, 1\}$ = the set consisting of 0 and 1,

$\underline{\underline{K}}$ = the <u>Cantor set</u>.

(c) $\overline{\underline{\underline{R}}} = [-\infty, \infty]$ (the <u>extended real line</u> with the usual topology).

This is homeomorphic to $\underline{\underline{I}}$ under the map $f : \underline{\underline{I}} \to \overline{\underline{\underline{R}}}$ defined by

$f(x) = \tan(\pi x - \frac{\pi}{2})$ $(0 < x < 1)$, $f(0) = -\infty$ and $f(1) = \infty$.

(d) $\underline{\underline{C}}$ = the complex numbers with the usual topology (the <u>complex plane</u>).

$\underline{\underline{C}}^n$ is called the <u>complex n-space</u> for $n < \infty$ and $\underline{\underline{C}}^\infty$ is called the <u>complex sequence space</u>.

(e) $\underline{\underline{2}}^\infty$ = the topological product of a countably infinite number of copies of $\underline{\underline{2}}$.

The following sets form a countable open base in $\underline{\underline{2}}^{\infty}$:

$$\underline{\underline{2}}_{j_1 j_2 \ldots j_r} = \{(i_1, i_2, \ldots) \in \underline{\underline{2}}^{\infty} : i_k = j_k,\ k=1,2,\ldots,r\},$$

$$r=1,2,\ldots\ ;\ j_k=0,1.$$

$\underline{\underline{2}}^{\infty}$ is homeomorphic to the Cantor set $\underline{\underline{K}}$ under the correspondence:

$$(i_1, i_2, \ldots) \longleftrightarrow \sum_{k=1}^{\infty} 2i_k \cdot 3^{-k}$$

(f) $\underline{\underline{N}}^{\infty}$ = the topological product of a countably infinite number of copies of $\underline{\underline{N}}$.

The following sets form a countable open base in $\underline{\underline{N}}^{\infty}$:

$$\underline{\underline{N}}_{m_1 m_2 \ldots m_r} = \{(n_1, n_2, \ldots) \in \underline{\underline{N}}^{\infty} : n_k = m_k,\ k=1,2,\ldots,r\}$$

$$r, j_k = 1,2,3,\ldots\ .$$

$\underline{\underline{N}}^{\infty}$ is homeomorphic to $\underline{\underline{J}} \cap \underline{\underline{I}}$ under the correspondence:

$$(n_1, n_2, \ldots) \longleftrightarrow \overline{n_1|}\ \overline{n_2|}\ \ldots \text{ (continued fraction).}$$

Let $\{r_n\}_{n\in\underline{\underline{Z}}}$ be a strictly increasing two-sided sequence of rational numbers such that

$$\lim_{n\to\infty} r_n = 1 \quad\text{and}\quad \lim_{n\to-\infty} r_n = 0 .$$

Defining $f(n)=r_n$ for $n \in \underline{\underline{Z}}$ and interpolating it linearly in each interval $(n, n+1)$, we obtain a bicontinuous map f : $\underline{\underline{R}} \to (0,1)$ such that

$$f(\underline{\underline{J}}) = \underline{\underline{J}} \cap \underline{\underline{I}} .$$

$\underline{\underline{J}}$ is homeomorphic to $\underline{\underline{J}} \cap \underline{\underline{I}}$ under a restriction of f . Therefore

$$\underline{\underline{J}} \approx \underline{\underline{J}} \cap \underline{\underline{I}} \approx \underline{\underline{N}}^{\infty} .$$

(g) $C[0,1]$ = the real continuous functions on $[0,1]$ with the maximum norm topology;

see § 2.11 .

(h) $D[0,1]$ = the real functions on $[0,1]$ continuous except for the discontinuities of the first kind with the Skorohod topology;
see § 2.12 .

(i) $L^0[0,1]$ = the Lebesgue measurable real functions on $[0,1]$ with the topology of convergence in measure;
see § 2.14 .

(j) $L^p[0,1]$ = the p-th order integrable real functions on $[0,1]$ with the p-th order norm topology, where $1 \leq p < \infty$;
see § 1.13 .

(k) $\mathcal{D}'(a)$ = the Schwartz distributions on $[-a,a]$,
$\mathcal{D}'$ = the Schwartz distributions on $\underline{\underline{R}}$;
see § 2.15 for the Schwartz topology in $\mathcal{D}'(a)$ or $\mathcal{D}'$.

1.5. Coincidence sets and diagonal sets.

Let f and g be maps from S into T. The set

$$C = C_{f,g} = \{x \in S : f(x) = g(x)\}$$

is called the coincidence set of f and g.

Theorem 1.5.1. Let S and T be topological spaces and suppose that T is Hausdorff. If $f : S \longrightarrow T$ and $g : S \longrightarrow T$ are continuous, then the coincidence set $C = C_{f,g}$ is closed in S.

Proof. It is enough to prove that every $x \in C^c$ has a neighborhood $U \subset C^c$. Since $x \in C^c$, $f(x) \neq g(x)$. Therefore we have disjoint neighborhoods $V = V(f(x))$ and $W = W(g(x))$, since T is Hausdorff. Then $U := f^{-1}(V) \cap g^{-1}(W)$ is a neighborhood of x by continuity of f and g. For every $y \in U$ we have

$$f(y) \in V \quad \text{and} \quad g(y) \in W$$

and hence $f(y) \neq g(y)$, i.e. $y \in C^c$.
This proves that $U \subset C^c$.

Let $\{S_i\}_{i \in I}$ be a family of subspaces of a Hausdorff space S and Π be the topological product $\prod_{i \in I} S_i$. The set of all $x \in \Pi$ with all components equal to each other is called the diagonal set of Π, denoted by $\Delta(\Pi)$. Let $p_i : \Pi \to S_i$ be the canonical projection and $e_i : S_i \to S$ be the canonical injection. Then

(1) $\Delta(\Pi) = \bigcap_{i,j \in I} \{\xi \in \Pi : (e_i \circ p_i)(\xi) = (e_j \circ p_j)(\xi)\}$.

Since p_i and e_i are continuous, we can use Theorem 1.5.1. to conclude that $\Delta(\Pi)$ is closed in Π.

Let D denote the intersection $\bigcap_{i\in I} S_i$. We will note that

(2) $$\Delta(\Pi) = \Delta(D^I) .$$

It is obvious that these diagonal sets are identical as sets. The topology on $\Delta(\Pi)$ is the relative topology as a subset of Π and the topology on $\Delta(D^I)$ is the relative topology as a subset of D^I. It is easy to check that these relative topologies are the same. Therefore the diagonal sets in (2) are identical as topological spaces.

Let us prove that $\Delta = \Delta(D^I)$ is homeomorphic to D under the map

$$f : D \longrightarrow \Delta$$

$$x \longmapsto (x)_{i\in I} := \text{the point with all components} = x .$$

It is obvious that f is bijective. Let q_i denote the canonical projection from D^I onto D carrying $(x_j)_{j\in I}$ to its i-component x_i. Then it is easy to see that

$$f^{-1} = q_i|_{\Delta} .$$

This implies that f^{-1} is continuous. Let $x \in D$ and V be any neighborhood of $f(x) = (x)_{i\in I}$ in Δ. Then we can find neighborhoods $U_k = U_k(x)$ in D such that

$$V \supset \bigcap_{k=1}^{n} q_{i_k}^{-1}(U_k) .$$

Let $U = \bigcap_{k=1}^{n} U_k$. Then U is a neighborhood of x in D and

$$V \supset \bigcap_{k=1}^{n} q_{i_k}^{-1}(U) \cap \Delta = f(U) .$$

This shows that f is continuous. Therefore $f : D \to \Delta$ is bicontinuous and $\Delta \approx D \quad (f)$.

Summarizing the results obtained above, we have the following.

<u>Theorem 1.5.2</u>. Let $\{S_i\}_{i \in I}$ be a family of subspaces of a Hausdorff topological space S. Then the diagonal set $\Delta(\prod_i S_i)$ is closed in $\prod_i S_i$ and is homeomorphic to the intersection $\bigcap_i S_i$.

1.6. Disjoint sums and projective limits.

Let $\{S_i\}_{i\in I}$ be a family of topological spaces. For each S_i we consider the set $S_i' = \{(x,i) : x \in S_i\}$ and topologize S_i' in the obvious way so that S_i' is homeomorphic to S_i under the map $x \mapsto (x,i)$. Thus we obtain a disjoint family $\{S_i'\}_{i\in I}$. Then the union $S' = \sum_{i\in I} S_i'$ is a topological space with the disjoint sum topology:

$$\mathcal{O}(S') = \{G' \subset S' : G' \cap S_i' \in \mathcal{O}(S_i') \text{ for every } i \in I\}.$$

The topological space S' is called the (topological) disjoint sum of $\{S_i\}_{i\in I}$, $\bigoplus_{i\in I} S_i$ in notation.

Theorem 1.6.1. Let $\{S_i\}_{i\in I}$ be a family of subspaces of a topological space S. Then the canonical map

$$J : \bigoplus_{i\in I} S_i \longrightarrow \bigcup_{i\in I} S_i \ (\subset S)$$

$$(x, i) \longmapsto x \ (i\in I,\ x\in S_i)$$

is a continuous surjection.

Proof. Obvious by the definition of $\mathcal{O}(S')$.

Let $\{S_i\}_{i\in I}$ be a family of topological spaces, where I is a directed set. Suppose that we are given a family Φ of continuous maps $\varphi_{ij} : S_j \longrightarrow S_i$, $i \leq j$, satisfying the following conditions:

(Φ. 1) $\varphi_{ii} : S_i \to S_i$ is the identity map,

(Φ. 2) $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$ for $i \leq j \leq k$.

Let

$$S = \{x \in \prod_{i\in I} S_i : p_i(x) = (\varphi_{ij} \circ p_j)(x) \text{ for } i \leq j\}.$$

Since S is a subset of the topological product $\prod_i S_i$, it is a topological space with the relative topology. The topological space S is called the projective limit of $\{S_i\}_{i\in I}$ relative to Φ, $\varprojlim_{\Phi} S_i$ in notation. Using Theorem 1.5.1., we can prove the following.

<u>Theorem 1.6.2.</u> If all spaces S_i, $i \in I$ are Hausdorff spaces, then $\varprojlim_{\Phi} S_i$ is a Hausdorff space and is closed in $\prod_i S_i$.

1.7. The analytic operation.

It is obvious that the sets $\underline{\underline{N}}^k$, $k=1,2,\ldots$, are disjoint. ?
Denote the union $\sum_{1 \le k < \infty} \underline{\underline{N}}^k$ by $\tilde{\underline{\underline{N}}}$. Elements of $\underline{\underline{N}}^\infty$ are denoted by $\underline{\underline{\ell}}$, $\underline{\underline{m}}$, $\underline{\underline{n}}$ or $\underline{\underline{\ell}}_i$, $\underline{\underline{m}}_i$, $\underline{\underline{n}}_i$ and their k-th components by ℓ_k, m_k, n_k or ℓ_{ik}, m_{ik}, n_{ik} respectively.

A system $\mathcal{S}$ of subsets of a set S indexed by the elements of $\tilde{\underline{\underline{N}}}$ is called a <u>Souslin scheme</u>. It is expressed as

$$\mathcal{S} = \{ A_{n_1 n_2 \ldots n_k} \},$$

where k and n_i move over $\underline{\underline{N}}$. With this Souslin scheme we associate a subset $K(\mathcal{S})$ of S, called the <u>kernel</u> of $\mathcal{S}$:

$$K(\mathcal{S}) = \bigcup_{\underline{\underline{n}} \in \underline{\underline{N}}^\infty} \bigcap_{k=1}^{\infty} A_{n_1 n_2 \ldots n_k}.$$

The operation:
$\mathcal{S} \longmapsto K(\mathcal{S})$ is called the <u>analytic operation</u>. Countable unions and countable intersections are special cases of the analytic operation:

$$\bigcup_n B_n = K(\mathcal{S}) \quad \text{for} \quad \mathcal{S} = \{ A_{n_1 n_2 \ldots n_k} := B_{n_1} \},$$

$$\bigcap_n B_n = K(\mathcal{S}) \quad \text{for} \quad \mathcal{S} = \{ A_{n_1 n_2 \ldots n_k} := B_k \}.$$

Let $\mathfrak{A}$ be a class of subsets of S. The class of all subsets obtained from sets in $\mathfrak{A}$ by the analytic operation is denoted by $\alpha[\mathfrak{A}]$. If $A \in \mathfrak{A}$, then

$$A = A \cup A \cup \ldots \in \alpha[\mathfrak{A}].$$

Therefore

$$\mathfrak{A} \subset \alpha[\mathfrak{A}].$$

<u>Theorem 1.7.1.</u> If $\mathfrak{A}$ is multiplicative, then
$\alpha[\alpha[\mathfrak{A}]] = \alpha[\mathfrak{A}] \supset \mathfrak{A}$.

Proof. By the last remark we always have $\mathfrak{A} \subset \alpha[\mathfrak{A}]$ and hence $\alpha[\mathfrak{A}] \subset \alpha[\alpha[\mathfrak{A}]]$. Therefore it is enough to prove that $\alpha[\alpha[\mathfrak{A}]] \subset \alpha[\mathfrak{A}]$, i.e. that for any Souslin scheme $\mathcal{S} = \{B_{n_1 n_2 \dots n_k}\}$ composed of sets in $\alpha[\mathfrak{A}]$, we have $K(\mathcal{S}) \in \alpha[\mathfrak{A}]$. Let

$$B_{n_1 n_2 \dots n_k} = \bigcup_{\underline{\underline{m}} \in \underline{\underline{N}}^\infty} \bigcap_{r=1}^{\infty} A^{n_1\, n_2\, \dots\, n_k}_{m_1\, m_2\, \dots\, m_r},$$

where all sets $A^{\cdots}_{\cdots}$ belong to $\mathfrak{A}$. Then

$$K(\mathcal{S}) = \bigcup_{\underline{\underline{n}} \in \underline{\underline{N}}^\infty} \bigcap_{k=1}^{\infty} \bigcup_{\underline{\underline{m}} \in \underline{\underline{N}}^\infty} \bigcap_{r=1}^{\infty} A^{n_1\, n_2\, \dots\, n_k}_{m_1\, m_2\, \dots\, m_r}.$$

Using the general distributive law of set theory, we can exchange $\bigcap_{k=1}^{\infty}$ and $\bigcup_{\underline{\underline{m}} \in N^\infty}$ to obtain

$$K(\mathcal{S}) = \bigcup_{\underline{\underline{n}} \in \underline{\underline{N}}^\infty} \bigcup_{\substack{\underline{\underline{m}}_i \in \underline{\underline{N}}^\infty \\ (i=1,2,\dots)}} \bigcap_{k=1}^{\infty} \bigcap_{r=1}^{\infty} A^{n_1\, n_2\, \dots\, n_k}_{m_{k1}\, m_{k2}\, \dots\, m_{kr}}$$

$$= \bigcup_{n,\, m_1, m_2, \dots \in \underline{\underline{N}}^\infty} \bigcap_{k,r=1}^{\infty} A^{n_1\, n_2\, \dots\, n_k}_{m_{k1}\, m_{k2}\, \dots\, m_{kr}}$$

For $\underline{\underline{n}}, \underline{\underline{m}}_1, \underline{\underline{m}}_2, \dots \in \underline{\underline{N}}^\infty$ fixed, the above intersection can be expressed as follows:

$$(1) \qquad \bigcap_{q=1}^{\infty} \bigcap_{k,r\,:\,k+r=q+1} A^{n_1\, n_2\, \dots\, n_k}_{m_{k1}\, m_{k2}\, \dots\, m_{kr}}.$$

Consider the indices $n_.$ and $m_{..}$ appearing in the inner intersection of (1). Arrange them in a triangular array and then divide them into q parts as shown in the diagram.

$$
\begin{array}{cc|c|c|c|c|}
n_1 & m_{11} & m_{12} & m_{13} & \cdots & m_{1q} \\
\cline{1-2}
 & n_2 & m_{21} & m_{22} & \cdots & m_{2,q-1} \\
\cline{2-3}
 & & n_3 & m_{31} & \cdots & m_{3,q-2} \\
\cline{3-4}
 & & & & \cdots & \\
\cline{5-5}
 & & & & n_q & m_{q1} \\
\cline{5-6}
\end{array}
$$

Let the inner intersection of (1) be denoted by

$$C(n_1\ m_{11} \mid n_2\ m_{12}\ m_{21} \mid n_3\ m_{13}\ m_{22}\ m_{31} \mid \cdots \mid n_q\ m_{1q}\ m_{2,q-1} \cdots m_{q1}) .$$

Then

$$K(\mathcal{S}) = \bigcup_{\underline{\underline{n}},\ \underline{\underline{m}}_1, \underline{\underline{m}}_2, \ldots \in \underline{\underline{N}}^\infty} \ \bigcap_{q=1}^{\infty} C(n_1\ m_{11} \mid \cdots \mid n_q\ m_{1q}\ m_{2,q-1} \cdots m_{q1}) .$$

Since the map

$$(\underline{\underline{n}},\ \underline{\underline{m}}_1,\ \underline{\underline{m}}_2,\ \ldots) \longmapsto C(n_1\ m_{11} \mid n_2\ m_{12}\ m_{21} \mid \cdots \mid n_q\ m_{1q} \cdots m_{q1} \mid \cdots)$$

gives a bijection:

$$\underline{\underline{N}}^\infty \times \underline{\underline{N}}^\infty \times \underline{\underline{N}}^\infty \times \ldots \longrightarrow \underline{\underline{N}}^2 \times \underline{\underline{N}}^3 \times \ldots \times \underline{\underline{N}}^{p+1} \times \ldots ,$$

we have

$$K(\mathcal{S}) = \bigcup_{\substack{(n_p, m_{1p}, \ldots, m_{p1}) \in \underline{\underline{N}}^{p+1} \\ (p=1,2,\ldots)}} \ \bigcap_{q=1}^{\infty} C(n_1\ m_{11} \mid \cdots \mid n_q\ m_{1q} \cdots m_{q1})$$

Since there exists a bijection $f_q : \underline{\underline{N}} \longrightarrow \underline{\underline{N}}^{q+1}$ for every q, we have

$$K(\mathcal{S}) = \bigcup_{\underline{\underline{\ell}} \in \underline{\underline{N}}^\infty} \ \bigcap_{q=1}^{\infty} C(f_1(\ell_1) \mid f_2(\ell_2) \mid \ldots \mid f_q(\ell_q)) .$$

Since $\mathfrak{A}$ is multiplicative, all sets $C(\cdots)$ belong to $\mathfrak{A}$. Therefore $K(\mathcal{S}) \in \alpha[\mathfrak{A}]$.

A Souslin scheme $\mathcal{S} = \{A_{n_1 n_2 \ldots n_k}\}$ is called <u>decreasing</u>, if $\{A_{n_1 n_2 \ldots n_k}\}_{k=1,2,\ldots}$ is decreasing for every $\underline{\underline{n}} = (n_k) \in \underline{\underline{N}}^\infty$. $\mathcal{S}$ is called <u>disjoint</u>, if $\{A_{n_1 n_2 \ldots n_{k-1}\, n}\}_{n=1,2,\ldots}$ is disjoint for every k and for every $(n_1, n_2, \ldots n_{k-1})$. (<u>Convention</u>: For $k=1$ this means that $\{A_n\}_n$ is decreasing.) If $\mathcal{S}$ is <u>regular</u> and ? ?
decreas.
disjoint, then the family

$$A_{n_1 n_2 \ldots n_k}, \quad (n_1, n_2, \ldots, n_k) \in \underline{\underline{N}}^k$$

is disjoint for every k.

Since a non-countable operation is involved in the analytic operation, $K(\mathcal{S}) \notin \sigma[\mathcal{S}]$ in general. But we have the following.

<u>Theorem 1.7.2.</u> For a decreasing disjoint Souslin scheme $\mathcal{S}$, we have

$$K(\mathcal{S}) = \bigcap_{k=1}^{\infty} \bigcup_{(n_1, n_2, \ldots, n_k) \in \underline{\underline{N}}^k} A_{n_1 n_2 \ldots n_k}$$

and therefore

$$K(\mathcal{S}) \in \sigma[\mathcal{S}].$$

<u>Proof.</u> Using the general distributive law of set theory, we can express the right hand side R as follows:

$$R = \bigcup_{\substack{(n_{k1}, n_{k2}, \ldots, n_{kk}) \in \underline{\underline{N}}^k \\ (k=1,2,\ldots)}} A_{n_{11}} \cap A_{n_{21} n_{22}} \cap A_{n_{31} n_{32} n_{33}} \cap \cdots .$$

Since $\mathcal{S}$ is decreasing and disjoint, all these countable intersections are empty except for

$$\begin{aligned} n_{11} = n_{21} = n_{31} &= \ldots \; (=n_1) \\ n_{22} = n_{32} &= \ldots \; (=n_2) \\ n_{33} &= \ldots \; (=n_3) \\ &\ldots \end{aligned}$$

Therefore

$$R = \bigcup_{(n_1,n_2,\ldots) \in \underline{\underline{N}}^\infty} A_{n_1} \cap A_{n_1 n_2} \cap A_{n_1 n_2 n_3} \cap \cdots = K(\mathcal{S}) .$$

Theorem 1.7.3. Let $\mathcal{B}$ be a σ-algebra on a set S. For any disjoint Souslin scheme $\mathcal{S} \subset \mathcal{B}$, we have

$$K(\mathcal{S}) \in \mathcal{B} .$$

Proof. Consider the Souslin scheme

$$\mathcal{S}' : A'_{n_1 n_2 \ldots n_k} = \bigcap_{i=1}^{k} A_{n_1 n_2 \ldots n_i} \in \mathcal{B} .$$

Then $\mathcal{S}'$ is decreasing disjoint Souslin scheme $\subset \mathcal{B}$. Since $K(\mathcal{S}) = K(\mathcal{S}')$ and since $K(\mathcal{S}') \in \sigma[\mathcal{S}'] \subset \mathcal{B}$ by the above theorem, we have $K(\mathcal{S}) \in \mathcal{B}$.

Let f be a map from a set S into another set T. For a Souslin scheme $\mathcal{S} = \{B_{n_1 n_2 \ldots n_k}\}$ of subsets of T the inverse image of $\mathcal{S}$ under f:

$$f^{-1}(\mathcal{S}) := \{f^{-1}(B_{n_1 n_2 \ldots n_k})\}$$

is a Souslin scheme of subsets of S. Since

$$f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n) \quad \text{and} \quad f^{-1}(\bigcap_n A_n) = \bigcap_n f^{-1}(A_n) ,$$

we have

$$K(f^{-1}(\mathcal{S})) = f^{-1}(K(\mathcal{S})).$$

Theorem 1.7.4. Let f be a map from a set S into another set T. Then

$$\alpha[f^{-1}(\mathcal{C})] = f^{-1}(\alpha[\mathcal{C}]) \qquad \text{for every } \mathcal{C} \subset 2^T .$$

1.8. Measures.

Let S be a set. A map α from a class $\mathfrak{A}$ of subsets of S into $\underline{\underline{R}}$, $\overline{\underline{\underline{R}}}$ or $\underline{\underline{C}}$ is called a <u>set function</u> on S and the class $\mathfrak{A}$ is called the <u>domain</u> of α, $\mathfrak{D}(\alpha)$ in notation.

A set function μ on S is called a measure on S if it satisfies the following conditions:

(μ,1) $\mathfrak{D}(\mu)$ is a σ-algebra on S,

(μ,2) $0 \leq \mu(A) \leq \infty$ for $A \in \mathfrak{D}(\mu)$, and $\mu(\phi) = 0$,

(μ,3) μ is <u>σ-additive</u>:

$$\mu\left(\sum_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n) \quad \text{for } \{A_n\} \subset \mathfrak{D}(\mu) \text{ disjoint.}$$

Let μ be a measure on S. A set $A \subset S$ is called <u>μ-measurable</u> if $A \in \mathfrak{D}(\mu)$. The value $\mu(A)$ for a given μ-measurable set A is called the <u>μ-measure</u> of A.

A measure μ on S is called

a <u>probability measure</u> (or a <u>stochastic measure</u>) if $\mu(S) = 1$,

a <u>substochastic measure</u> if $\mu(S) \leq 1$,

a <u>finite measure</u> if $\mu(S) < \infty$,

and

a <u>σ-finite measure</u> if we have a sequence $\{S_n\} \subset \mathfrak{D}(\mu)$ such that

$S = \bigcup_n S_n$ and $\mu(S_n) < \infty$ for every n.

A measure μ on S is called <u>complete</u>, if

$\mu(A) = 0,\ B \subset A \Rightarrow B \in \mathfrak{D}(\mu)$ (and hence $\mu(B)=0$).

For a measure μ on S we define two set functions with domain 2^S:

the outer μ-measure $\mu^*(A) := \inf\{\mu(B) : B \in \mathfrak{D}(\mu),\ B \supset A\}$,

the inner μ-measure $\mu_*(A) := \sup\{\mu(B) : B \in \mathfrak{D}(\mu),\ B \subset A\}$.

For every $A \subset S$, we can find $B_1, B_2 \in \mathfrak{D}(\mu)$ such that

(1) $\quad B_1 \subset A \subset B_2$ and $\mu(B_1) = \mu_*(A) \le \mu^*(A) = \mu(B_2)$.

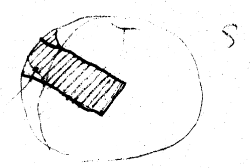
All properties of μ^* and μ_* can be derived from this.

Let ν be a measure on S and let $\mathcal{B}$ be a σ-algebra on S included in $\mathfrak{D}(\nu)$. Then the restriction $\mu = \nu|_{\mathcal{B}}$ is a measure on S. A measure ν is called an <u>extension</u> of a measure μ, if $\mathfrak{D}(\nu) \supset \mathfrak{D}(\mu)$ and $\nu = \mu$ on $\mathfrak{D}(\mu)$. A complete measure which is an extension of μ is called a <u>complete extension</u> of μ. There are many complete extensions. The minimum complete extension of μ is called the <u>Lebesgue extension</u> of μ, denoted by $\bar{\mu}$. It is defined as follows:

$$\mathfrak{D}(\bar{\mu}) = \{A \subset S : B_1 \subset A \subset B_2 \text{ for some } B_1, B_2 \in \mathfrak{D}(\mu) \text{ with } \mu(B_2 - B_1) = 0\}$$

$$\bar{\mu}(A) = \mu^*(A) \ (= \mu_*(A)) \quad \text{for } A \in \mathfrak{D}(\bar{\mu}).$$

Let μ be a measure on S and let T be a subset of S (μ-measurable or not) such that $\mu_*(S-T) = 0$. Define a set function ν on S by



$$\mathfrak{D}(\nu) = \{A \subset S : A \cap T \in \mathfrak{D}(\mu) \cap T\},$$

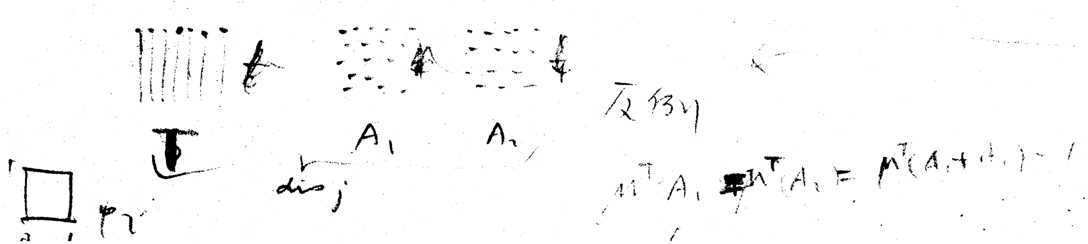
$$\nu(A) = \mu^*(A).$$

?

Using the assumption $\mu_*(S-T) = 0$, we can check that ν is a measure on S extending μ. The measure ν is denoted by μ^T. The domain $\mathfrak{D}(\nu)$ is the σ-algebra generated by T and all sets in $\mathfrak{D}(\mu)$, i.e.

$$\mathfrak{D}(\nu) = \sigma[\{T\} \cup \mathfrak{D}(\mu)].$$

If T is μ-measurable, then $\mu^T = \mu$. Otherwise μ^T is a strict extension of μ. If μ is complete, then μ^T is also complete.



Let μ be a measure on S. For a subset T of S (μ-measurable or not) we define a set function θ on T by

$$\mathfrak{D}(\theta) = \mathfrak{D}(\mu) \cap T$$

$$\theta(A) = \mu^*(A) \qquad \text{for } A \in \mathfrak{D}(\theta) .$$

Then θ is a measure on T. The measure θ is called the <u>trace measure</u> of μ on T, μ_T in notation. If μ is complete, then μ_T is also complete. If T is μ-measurable, then $\mathfrak{D}(\mu) \cap T \subset \mathfrak{D}(\mu)$ and $\mu_T = \mu\big|_{\mathfrak{D}(\mu) \cap T}$.

In this case μ_T is often denoted by $\mu\big|_T$.

Note that μ_T is a measure on T, while μ^T is a measure on S.

A set S endowed with a measure μ is called a <u>measure space</u>, denoted by $S(\mu)$ or (S, μ).

Let f be a map from a measure space $S(\mu)$ into a Borel space $T(\mathcal{T})$ ($\mathcal{T} = \mathcal{B}(T)$ if T is a topological space). The map f is called <u>μ-measurable</u> if f is measurable $\mathfrak{D}(\mu)/\mathcal{T}$.

On a measure space $S(\mu)$ we can define the <u>integral</u> of a μ-measurable function $f : S(\mu) \to \underline{\underline{R}}$ (or $\overline{\underline{\underline{R}}}$ or $\underline{\underline{C}}$) on a μ-measurable set A, denoted by

$$\int_A f(x)\,\mu(dx) \quad \text{or} \quad \int_A f\,d\mu ,$$

under certain conditions. We assume the reader to be familiar with fundamental facts in the theory of measures and integrals.

Let f be a map from a set S into another set T. For a measure μ on S we define the <u>image measure</u> of μ under f, denoted by $f\mu$ as follows:

$$\mathfrak{D}(f\mu) = \{B \subset T : f^{-1}(B) \in \mathfrak{D}(\mu)\} ,$$

$$f\mu(B) = \mu(f^{-1}(B)) .$$

The <u>transformation formula</u> on integrals:

$$\int_B g(y)\, f\mu\widehat{\ }(dy) = \int_{f^{-1}(B)} (g\circ f)(x)\, \mu(dx)$$

holds in the sense that if one of these integrals is well-defined, then the other is well-defined and has the same value.

If the original measure μ on S is complete, then the image measure $f\mu$ on T is also complete. It is obvious that

$$f\mu(T) = \mu(S) .$$

Therefore if μ is stochastic, then $f\mu$ is stochastic. Similarly for substochastic or finite measures. However, even if μ is σ-finite, $f\mu$ is not always σ-finite; for example,

$S = \underline{\underline{R}}^2$, $T = \underline{\underline{R}}$, μ = the Lebesgue measure on R^2

f = the canonical projection : $(x, y) \mapsto x$.

Since the domain of a measure is a σ-algebra, it is closed under countable operations such as countable unions, countable intersections, and so on, but it is not always closed under the analytic operation. However, we have the following important theorem.

<u>Theorem 1.8.1.</u> Let μ be a σ-finite complete measure on S . Then $\mathcal{D}(\mu)$ is closed under the analytic operation.

<u>Proof</u>. Let $\mathcal{S} = \{A_{n_1 n_2 \dots n_k}\}$ be a Souslin scheme composed of μ-measurable sets. We will prove that $K(\mathcal{S})$ is μ-measurable. Since μ is σ-finite, we have

$$S = \bigcup_m S_m, \quad \mu(S_m) < \infty , \quad m=1,2,\dots .$$

Since

$$\mathcal{S}^m : A^m_{n_1 n_2 \dots n_k} := \bigcap_{i=1}^{k} A_{n_1 n_2 \dots n_i} \cap S_m$$

is a decreasing Souslin scheme composed of μ-measurable subsets of S_m and since

$$K(\mathcal{S}) = \bigcup_m K(\mathcal{S}^m) \ ,$$

it is enough to prove that $K(\mathcal{S}^m)$ is μ-measurable for every m. Therefore we can assume without loss of generality that $\mathcal{S}$ is a decreasing Souslin scheme composed of μ-measurable subsets of S', where $\mu(S') < \infty$.

Define two Souslin schemes:

$$\overline{\mathcal{S}} : \overline{A}_{n_1 n_2 \ldots n_k} := \bigcup_{h_i \leq n_i \ (i=1,2,\ldots k)} A_{h_1 h_2 \ldots h_k} \ ,$$

$$\underline{\mathcal{S}} : \underline{A}_{n_1 n_2 \ldots n_k} := \bigcup_{\substack{h_i \leq n_i \ (i=1,2,\ldots,k) \\ h_i \in \underline{\underline{N}} \ (i > k)}} \bigcap_{j=1}^{\infty} A_{h_1 h_2 \ldots h_j} \ .$$

Then

(1) $\overline{\mathcal{S}}$ and $\underline{\mathcal{S}}$ are decreasing Souslin schemes,

(2) $\underline{A}_{n_1 n_2 \ldots n_k} \subset \overline{A}_{n_1 n_2 \ldots n_k}$,

(3) $\overline{A}_{n_1 n_2 \ldots n_k} \in \mathfrak{D}(\mu)$;

note that $\underline{A}_{n_1 n_2 \ldots n_k} \notin \mathfrak{D}(\mu)$ in general.

First we will prove that

(4) $\bigcap_k \overline{A}_{n_1 n_2 \ldots n_k} \subset K := K(\mathcal{S})$ for every $(n_k) \in \underline{\underline{N}}^{\infty}$.

Let x be any element of the intersection. Then we can find a triangular array of indices:

$$\begin{array}{llll} h_{11}, & h_{21}, & h_{31}, & \dots \leq n_1 \\ & h_{22}, & h_{32}, & \dots \leq n_2 \\ & & h_{33}, & \dots \leq n_3 \\ & & & \dots \end{array}$$

such that

$$x \in A_{h_{k1}h_{k2}\dots h_{kk}} \qquad \text{for } k=1,2,\dots$$

Since $h_{k1} \leq n_1$ for each k, we can find $r_1 \leq n_1$ such that

$$h_{k1} = r_1 \qquad \text{for infinitely many } k\text{'s} .$$

Observing h_{k2} for such k's, we can find $r_2 \leq n_2$ such that

$$h_{k1} = r_1 \text{ and } h_{k2} = r_2 \quad \text{for infinitely many } k\text{'s} .$$

Repeating this, we can find a sequence $r_i \leq n_i$, $i=1,2,\dots$ such that for each i, we have

$$h_{k1} = r_1,\ h_{k2} = r_2,\ \dots,\ h_{ki} = r_i \quad \text{for infinitely many } k\text{'s} .$$

Taking, for each i, a number $k = k(i)$ satisfying the above conditions, we have

$$x \in A_{h_{k1}h_{k2}\dots h_{kk}} = A_{r_1r_2\dots r_i\, h_{k,i+1}\dots h_{k,k}} \subset A_{r_1r_2\dots r_i}$$

for every i. Therefore

$$x \in \bigcap_i A_{r_1r_2\dots r_i} \subset K .$$

Since $\underline{A}_n \uparrow K \ (n \to \infty)$, we have

$$\mu^*(\underline{A}_n) \uparrow \mu^*(K) .$$

In fact, by taking $B_n \in \mathcal{B}(\mu)$ with $B_n \supset \underline{A}_n$ and $\mu(B_n) = \mu^*(\underline{A}_n)$, we have

$$\overline{\lim_n}\ \mu^*(\underline{A}_n) \le \mu^*(K) \le \mu(\underline{\lim}_n B_n) \le \underline{\lim}_n \mu(B_n) = \underline{\lim}_n \mu^*(\underline{A}_n) .$$

Similarly we have

$$\mu^*(\underline{A}_{n_1 n_2 \ldots n_k n}) \uparrow \mu^*(\underline{A}_{n_1 n_2 \ldots n_k}) \quad (n \to \infty) .$$

Therefore, for every $\varepsilon > 0$, we can find $m_1, m_2, \ldots$ such that

$$\begin{aligned} \mu^*(K) &< \mu^*(\underline{A}_{m_1}) + 2^{-1}\varepsilon \\ &< \mu^*(\underline{A}_{m_1 m_2}) + 2^{-2}\varepsilon + 2^{-1}\varepsilon \\ &\cdots\cdots \\ &< \mu^*(\underline{A}_{m_1 m_2 \ldots m_k}) + 2^{-k}\varepsilon + 2^{-(k-1)}\varepsilon + \cdots + 2^{-1}\varepsilon \\ &\cdots\cdots \end{aligned}$$

This implies that

$$\begin{aligned} \mu^*(K) &\le \lim_k \mu^*(\underline{A}_{m_1 m_2 \ldots m_k}) + \varepsilon \\ &\le \lim_k \mu(\overline{A}_{m_1 m_2 \ldots m_k}) + \varepsilon && \text{by (2) and (3)} \\ &= \mu(\bigcap_k \overline{A}_{m_1 m_2 \ldots m_k}) + \varepsilon && \text{by (1)} \\ &\le \mu_*(K) + \varepsilon && \text{by (4)} \end{aligned}$$

Letting $\varepsilon \downarrow 0$, we have

$$\mu^*(K) = \mu_*(K) \le \mu(S') < \infty .$$

Therefore we have $B_1, B_2 \in \mathcal{B}(\mu)$ such that $B_1 \subset K \subset B_2$ and

$$B_1 \subset K \subset B_2 \quad \text{and} \quad \mu(B_1) = \mu_*(K) = \mu^*(K) = \mu(B_2) ,$$

which implies that $\mu(B_2 - B_1) = 0$. Since μ is complete, we have $K \in \mathcal{B}(\mu)$.

1.9. B-regular measures and universally measurable sets.

Let $S(\mathcal{S})$ be a Borel space. A measure μ on S is called a B-regular measure on $S(\mathcal{S})$ if it satisfies the following conditions:

(B, 1) μ is complete,

(B, 2) $\mathfrak{D}(\mu) \supset \mathcal{S}$,

(B, 3) For every $A \in \mathfrak{D}(\mu)$ we can find $B \in \mathcal{S}$ such that

$$B \subset A \quad \text{and} \quad \mu(A-B) = 0 \ .$$

It is easy to check that (B, 3) is equivalent to the condition:

(B, 3') For every $A \in \mathfrak{D}(\mu)$ we can find $B_1, B_2 \in \mathcal{S}$ such that

$$B_1 \subset A \subset B_2 \quad \text{and} \quad \mu(B_2 - B_1) = 0 \ .$$

Therefore the following conditions are equivalent to each other.

(1) μ is B-regular,

(2) μ is the Lebesgue extension of a measure on S with domain $\mathcal{S}$, $\mathfrak{D}(\mu) \supset \mathcal{S}$ a.s.

(3) $\mu = \overline{(\mu|_{\mathcal{S}})}$.

We can define B-regular measures on a topological space by regarding the space as a Borel space with the topological σ-algebra.

Let $S(\mathcal{S})$ and $T(\mathcal{T})$ be Borel spaces, μ a B-regular measure and $f : S \rightarrow T$ a μ-measurable map. Then the image measure $f\mu$ is complete and $\mathfrak{D}(f\mu) \supset \mathcal{T}$. Therefore $f\mu$ is a complete extension of the restriction $f\mu|_{\mathcal{T}}$. This implies that $f\mu$ is an extension of $\overline{(f\mu|_{\mathcal{T}})}$. But $f\mu \neq \overline{(f\mu|_{\mathcal{T}})}$ in general. Indeed, $f\mu$ is not always B-regular even if f is Borel measurable (and so obviously μ-measurable), as the following trivial example shows. Let

$$S = \{a_1, a_2, \ldots, a_n\}, \quad \mathcal{S} = 2^S,$$

$$T = \{b_1, b_2, \ldots, b_n\}, \quad \mathcal{T} = \{\emptyset, T\},$$

μ = the counting measure on S.

and define $f : S \to T$ by $a_k \mapsto b_k$ $(k=1,2,\ldots,n)$. Then $\nu := f\mu$ is the counting measure with $\mathfrak{D}(\nu) = 2^T$. Therefore

$$\overline{(\nu|_{\mathcal{T}})} = \nu|_{\mathcal{T}} \neq \nu \qquad \text{for } n > 1 .$$

Let us give a more sophisticated example in which S and T are subspaces of $\underline{\underline{R}}$. Let λ be the (classical) Lebesgue measure on $\underline{\underline{I}} = [0,1]$. The measure λ is B-regular. Let

S = the famous non-λ-measurable subset of $\underline{\underline{I}}$ due to Lebesgue,

$T = \underline{\underline{I}}$,

$\mu = \lambda_S$ (the trace measure of λ on S)

and f = the canonical injection from S into T. Then μ is B-regular and f is Borel measurable (hence μ-measurable). Examining Lebesgue's construction of S, we have $\lambda^*(S) = 1$ and so $\lambda^*(A \cap S) = \lambda^*(A)$ for $A \subset \underline{\underline{I}}$. Therefore

$$f\mu(B) = \mu(f^{-1}(B)) = \mu(B \cap S) = \lambda^*(B \cap S) = \lambda(B) \quad \text{for } B \in \mathfrak{B}(T).$$

If $f\mu$ were B-regular, we would have

$$f\mu = \lambda ,$$

in contradiction to $S \in \mathfrak{D}(f\mu)$ and $S \notin \mathfrak{D}(\lambda)$.

A subset of a Borel space $S(\mathcal{S})$ is called universally measurable if it is measurable with respect to every B-regular probability measure. The class of all universally measurable subsets of $S(\mathcal{S})$ is a σ-algebra on $S(\mathcal{S})$ including $\mathcal{S}$ and is denoted by $\mathcal{M}_u(S, \mathcal{S})$ or $\mathcal{M}_u(S)$. A map $f : S(\mathcal{S}) \to T(\mathcal{T})$ is called universally measurable, if f is measurable $\mathfrak{M}_u(S)/\mathfrak{T}$. It is obvious that every Borel measurable k—map is universally measurable.

Theorem 1.9.1. Every universally measurable subset of $S(\mathcal{S})$ is measurable with respect to every σ-finite B-regular measure on $S(\mathcal{S})$.

Proof. Let A be universally measurable and let μ be a σ-finite B-regular measure on $S(\mathcal{S})$. Since μ is σ-finite, we can find a disjoint countable family $\{S_n\} \subset \mathfrak{D}(\mu)$ such that

$$S = \sum_n S_n, \quad 0 < \mu(S_n) < \infty \qquad (n=1,2,\ldots).$$

For each n, define a B-regular probability measure μ_n on $S(\mathcal{S})$ by

$$\mu_n(B) = \frac{\mu(B \cap S_n)}{\mu(S_n)} \qquad \text{for } B \in \mathcal{S}.$$

Then

$$\mu(B) = \sum_n \mu(S_n)\,\mu_n(B) \qquad \text{for } B \in \mathcal{S}.$$

Since A is μ-measurable, we can find $B_{n1}, B_{n2} \in \mathcal{S}$ such that

$$B_{n1} \subset A \subset B_{n2}, \quad \mu_n(B_{n2} - B_{n1}) = 0.$$

Let $B_1 = \bigcup_n B_{n1}$ and $B_2 = \bigcap_n B_{n2}$. Then

$$B_1, B_2 \in \mathcal{S}, \quad B_1 \subset A \subset B_2, \quad B_2 - B_1 \subset B_{n2} - B_{n1} \ (n=1,2,\ldots)$$

and

$$\mu(B_2 - B_1) = \sum_n \mu(S_n)\,\mu_n(B_2 - B_1) = 0.$$

Theorem 1.9.2. $\mathfrak{M}_u(S,\mathcal{S})$ is a σ-algebra on S including $\mathcal{S}$ and is closed under the analytic operation. Therefore $\mathfrak{M}_u(S,\mathcal{S}) \supset \alpha[\mathcal{S}]$.

Proof. Obvious by Theorem 1.8.1.

We define <u>universally measurable sets on a topological space</u>, regarding the space as a Borel space with the topological σ-algebra. Similarly we define <u>universally measurable maps from a topological (or Borel) space into another topological (or Borel) space.</u>

1.10. K-regular measures.

Let S be a topological space and μ an arbitrary measure on S (B-regular or not). A μ-measurable set A is called K-regular if

(1) $\mu(A) = \sup \{\mu(K) : K \text{ compact}, K \subset A, K \in \mathfrak{D}(\mu)\}$.

If μ is a B-regular measure and if every μ-measurable set is K-regular, then μ is called a K-regular measure on S.

A class of sets A_i, $i \in I$, is called directed up (resp. directed down) if for every $i, j \in I$ there exists $k \in I$ such that $A_k \supset A_i \cup A_j$ (resp. $A_k \subset A_i \cap A_j$) .

Theorem 1.10.1. Let μ be a K-regular measure on S .

(i) If $\{G_i\}_{i \in I}$ is a class of open sets directed up, then

$$\mu(\cup_i G_i) = \sup_i \mu(G_i) .$$

(ii) If $\{F_i\}_{i \in I}$ is a class of closed sets directed down and if $\mu(F_j) < \infty$ for some j , then

$$\mu(\cap_i F_i) = \inf_i \mu(F_i) .$$

Remark. If I is countable, this is obvious by the general properties of a measure.

Proof.

(i) Let $G = \cup_i G_i$ and let K be any compact set included in G . Since $\{G_i\}_i$ is directed up, we have $G_j \supset K$ for some j . Then

$$\sup_i \mu(G_i) \geq \mu(G_j) \geq \mu(K) .$$

Since μ is K-regular, we have

$$\sup_i \mu(G_i) \geq \mu(G) .$$

The opposite inequality is obvious.

(ii) Since $\{F_i\}_i$ is directed down, $\bigcap_i F_i$ and $\inf_i \mu(F_i)$ do not change, even if we ignore the sets that are not included in F_j. Therefore we can assume that $\bigcup_i F_i$ is included in a set S' with $\mu(S') < \infty$. Since $S'-F_i$ is open in S' for every i, we can use the same argument as in (i) to obtain

$$\mu(\bigcup_i (S'-F_i)) = \sup_i \mu(S'-F_i) .$$

Since $\mu(S') < \infty$, this implies the conclusion of (ii).

Let S and T be topological spaces and let μ be a K-regular measure on S. A μ-measurable (i.e. measurable $\mathcal{D}(\mu)/\mathcal{B}(T)$) map $f : S \to T$ is called Lusin μ-measurable if for every set $A \in \mathcal{D}(\mu)$ and every $a < \mu(A)$, we can find a compact set $K \subset A$ such that

(L) $\mu(K) > a$ and the restriction $f|_K$ is continuous.

Theorem 1.10.2. Let S and T be Hausdorff spaces, μ a σ-finite B-regular measure on S, and let $f : S \longrightarrow T$ be Lusin μ-measurable. The image measure $\nu = f\mu$ on T is K-regular.

Proof. First we will prove that every $\hat{\nu}$-measurable set B is K-regular. Let $a < \nu(B)$. Then

$$A := f^{-1}(B) \in \mathcal{D}(\mu) \quad \text{and} \quad \mu(A) = \mu(B) > a .$$

Since f is Lusin μ-measurable, we can find a compact set $K \subset A$ such that

$$\mu(K) > a \quad \text{and} \quad g = f|_K \quad \text{is continuous.}$$

Let $H = g(K)(=f(K))$. Then H is compact by continuity of f, $H \subset f(A) \subset B$, and

$$\nu(H) = \mu(f^{-1}(H)) \geq \mu(K) > a .$$

This proves that every $B \in \mathfrak{D}(\nu)$ is K-regular.

Since μ is complete, ν is also complete. It remains only to prove that for every $B \in \mathfrak{D}(\nu)$, we can find $C \in \mathcal{B}(T)$ such that $C \subset B$ and $\nu(B-C) = 0$. Since this property of B is inherited by countable unions and since ν is σ-finite, we can assume without loss of generality that $\nu(B) < \infty$. Since $B \in \mathfrak{D}(\nu)$ is K-regular, we can find a sequence of compact sets $H_n \subset B$, $n=1,2,\ldots$ such that $\nu(H_n) \uparrow \nu(B)$.
Let C denote the union $\bigcup_n H_n$. Since T is a Hausdorff space, H_n is closed and therefore $C \in \mathcal{B}(T)$. It is obvious that

$$C \subset B \quad \text{and} \quad \nu(C) = \nu(B) .$$

Since $\nu(B) < \infty$, we have $\nu(B-C) = 0$. This completes the proof of our theorem.

1.11. The weak topology in the space of measures.

A topological space S is called completely regular if it is Hausdorff and if for every point a of S and open subset G of S containing a, we can find a continuous function

$$f = f_{a,G} : S \to [0,1]$$

such that $f(a) = 1$ and $f(x) = 0$ for $x \in G^c$. We note that this condition implies that for every compact subset K of S and open subset G of S, containing K, we can find a continuous function

$$f = f_{K,G} : S \to [0,1]$$

such that $f = 1$ on K and $f = 0$ on G^c. To see this, choose $f_a = f_{a,G}$ for each $a \in K$. Then the family

$$U(a) = \left\{ x : f_a(x) > \tfrac{1}{2} \right\}, \quad a \in K,$$

is an open covering of K. Since K is conpact, we can find $a_1, a_2, \ldots, a_n$ K such that $\bigcup_i U(a_i) \supset K$. Then the function

$$f = \max_i (2f_{a_i} \wedge 1)$$

satisfies the condition for $f_{K,G}$.

In this section S always stands for a completely regular space. We denote by $C^+(S)$ and $\mathcal{M}^+(S)$ the bounded continuous non-negative functions on S and the finite K-regular measures on S respectively. The integral

$$\int_S f d\mu$$

is denoted by $\mu(F)$.

K If $\mu \in \mathcal{M}^+(S)$, then

(1) $\mu(A) = \sup \{ \mu(K) : K \subset A, K \text{ compact} \}$

and

(2) $\mu(A) = \inf \{ \mu(G) : G \supset A, G \text{ open} \}$

for $A \in \mathfrak{B}(\mu)$. Therefore any $\mu \in \mathcal{M}^+(S)$ is completely determined by its behavior on compact sets. Since

$$\mu(K) \leq \mu(f_{K,G}) \leq \mu(G)$$

for the $f_{K,G}$ mentiond above, we have

(3) $\quad \mu(K) = \inf \{ \mu(f) \; : \; f \in C^+(S),\ f \geq 1 \text{ on } K \}$,

which immediately implies

Theorem 1.11.1. Let $\mu_1, \mu_2 \in \mathcal{M}^+(S)$. If $\mu_1(f) = \mu_2(f)$ for evey $f \in C^+(S)$, then $\mu_1 = \mu_2$.

A functional on $C^+(S)$ is called <u>positive</u> if $\ell(f) \geq 0$ for every $f \in C^+(S)$, and <u>additive</u> if

$$\mu(f + g) = \mu(f) + \mu(g).$$

A positive additive functional ℓ on $C^+(S)$ is called tight if for every $\epsilon > 0$ there exists a compact set $K = K(\epsilon)$ such that

(4) $\quad f \geq 1 \text{ on } K \Rightarrow \ell(f) \geq \ell(1_S) - \epsilon$.

It is obvious that for $\mu \in \mathcal{M}^+(S)$ $\mu(f)$, regarded as a functional on $C^+(S)$, is positive, additive and tight; tightness follows from (1). conversely we have

Theorem 1.11.2. For every positive, additive and tight functional ℓ on $C^+(S)$ we can find a unique $\mu \in \mathcal{M}^+(S)$ such that

$$\ell(f) = \mu(f) \quad \text{for every } f \in C^+(S).$$

<u>Remark.</u> If S is a compact Hausdorff space, this theorem is well-known as the Riesz representation theorem. Note that tightness is automatic in this case, because we can take S for the compact set K in (4).

<u>Proof of the theorem</u>. To prove existence, Let S be the Stone-Čech compactification of S, which exists by complete regularity of S. Then the map

$$\varphi : C(\tilde{S}) \to C(S)$$
$$\tilde{f} \mapsto \tilde{f}|_S \quad \text{(the restriction of } f \text{ to } S)$$

is bijective. The functional $\tilde{\ell}$ on $C^+(\tilde{S})$ defined by

$$\tilde{\ell}(\tilde{f}) = \ell(\tilde{f}|_S)$$

is positive and additive. Since $\tilde{S}$ is a compact Hausdorff space, we can use the Riesz representation theorem to find $\tilde{\mu} \in \mathfrak{M}^+(\tilde{S})$ such that

$$\tilde{\ell}(\tilde{f}) = \tilde{\mu}(\tilde{f}) \quad \text{for} \quad \tilde{f} \in C^+(\tilde{S}) .$$

We will prove that

(5) $\quad S \in \mathfrak{D}(\tilde{\mu})$ and $\tilde{\mu}(\tilde{S} - S) = 0.$

For every $\varepsilon > 0$ we can find a compact set $K = K(\varepsilon)$ satisfying the condition (4). Then

$$\begin{aligned} \tilde{f} \geq 1 \text{ on } K &\Rightarrow \tilde{f}|_S \geq 1 \text{ on } K \\ &\Rightarrow \ell(f|_S) \geq \ell(1_S) - \varepsilon \\ &\Rightarrow \tilde{\ell}(f) \geq \tilde{\ell}(1_{\tilde{S}}) - \varepsilon = \tilde{\mu}(\tilde{S}) - \varepsilon. \end{aligned}$$

Hence we have

$$\tilde{\mu}(K) \geq \tilde{\mu}(\tilde{S}) - \varepsilon,$$

which implies (5) since $K \subset S \subset \tilde{S}$.

By virtue of (5) the restriction $\mu = \tilde{\mu}|_S$ is a finite K-regular measure on S. For every $f \in C^+(S)$ we have a unique $\tilde{f} \in C^+(S)$ such $f = \tilde{f}|_S$. Hence

$$\begin{aligned} \ell(f) &= \tilde{\ell}(\tilde{f}) = \tilde{\mu}(\tilde{f}) \\ &= \int_S \tilde{f}\, d\tilde{\mu} \quad (\text{since } \mu(\tilde{S} - S) = 0) \\ &= \int_S f\, d\mu = \mu(f), \end{aligned}$$

which completes the proof of the theorem.

The <u>weak topology</u> in $\mathfrak{M}^+(S)$ is induced by the neighborhoods

$$U_{f_1 f_2 \cdots f_n \varepsilon}(\mu) = \{ \nu \in \mathfrak{M}^+(S) : |\nu(f_i) - \mu(f_i)| < \varepsilon , \ i = 1,2,\ldots,n \}$$

$\mu \in \mathfrak{M}^+(S), f_1, f_2, \ldots, f_n \in C^+(S)$, $\quad \varepsilon > 0.$

It is easy to check that these give a well-defined T_1-topology; the separation axiom follows from Theorem 1.11.1.

Consider the map

$$(6) \qquad i : \mathcal{M}^+(S) \to \underline{\underline{R}}^{C^+(S)}$$
$$\mu \mapsto (\mu(f))_{f \in C^+(S)} .$$

This is injective by Theorem 1.11.1 and

$$i(U_{f_1 f_2 \dots f_n, \varepsilon}(\mu)) = p_{f_1}^{-1}(-\varepsilon,\varepsilon) \cap p_{f_2}^{-1}(-\varepsilon,\varepsilon) \cap \dots \cap p_{f_n}^{-1}(-\varepsilon,\varepsilon) \cap i(S).$$

where p_f is the canonical projection to the f-component. Hence the space $\mathcal{M}^+(S)$ with the weak topology is homeomorphic to the subspace $i(\mathcal{M}^+(S))$ of $\underline{\underline{R}}^{C^+(S)}$ with the product topology. As $i(\mathcal{M}^+(S))$ is completely regular, so is $\mathcal{M}^+(S)$. Thus we have

Theorem 1.11.3. The space $\mathcal{M}^+(S)$ with the weak topology is completely regular.

A subset M of a topological space S is called conditionally compact in S ~~of~~ when the closure of M in S ~~ins~~ is compact.

Theorem 1.11.4.
A subset M of $\mathcal{M}^+(S)$ is conditionally compact in $\mathcal{M}^+(S)$ if the following conditions are satisfied :

(i) (Uniform boundedness) $\sup_{\mu \in M} \mu(S) \leq A < \infty$,

(ii) (Uniform K-regularity) $\inf_{K \text{ compact}} \sup_{\mu \in M} \mu(K^c) = 0.$

Remark. The second condition is automatic if S is compact.

Proof of the theorem. Let Γ denote $\underline{\underline{R}}^{C^+(S)}$. By the map $i : \mathcal{M}^+(S) \to \Gamma$ in (6) we imbed $\mathcal{M}^+(S)$ into Γ. Theorem 1.11.2 shows that an element $\ell = (\ell(f),\ f \in C^+(S))$ of Γ belongs to $\mathcal{M}^+(S)$ if and only if $\ell(f)$, as a functional of f, is positive, additive and tight. The closure $\bar{M}$ of M in $\mathfrak{M}^+(S)$ is $M^* \cap \mathfrak{M}^+(S)$, where M^* is the closure of M in Γ. But we can prove that

(6) $\bar{M} = M^*$.

To do this, it is enough to show that

(7) $\ell_0 \in M^* \Rightarrow \ell_0 \in \mathfrak{M}^+(S)$.

By the assumption $\ell_0 \in M^*$ we can therefore find a generalized sequence $\mu_\alpha \in M$ converging to ℓ_0 in Γ. Since $\{\mu_\alpha\} \subset \mathfrak{M}^+(S)$ and since the map $\ell \to \ell(f)$ is continuous for every $f \in C^+(S)$, $\ell_0(f)$, as a functional of f, is positive and additive. By ~~the~~ condition (ii) we can find a compact set $K = K(\varepsilon)$ $(\varepsilon > 0)$ such that

$$\mu_\alpha(K^c) < \varepsilon \qquad \text{for every } \alpha.$$

Let f be any function in $C^+(S)$ such that $f \geq 1$ on K. Then we have

$$\mu_\alpha(f) \geq \mu_\alpha(1_S) - \varepsilon \qquad \text{for every } \alpha,$$

so

$$\ell_0(f) \geq \ell_0(1_S) - \varepsilon$$

~~by~~ since $\mu_\alpha \to \ell_0$. This proves that $\ell_0(f)$ is positive, additive and tight, i.e. $\ell_0 \in \mathfrak{M}^+(S)$. Thus we have proved (7).

Let $a = \sup_M \mu(S)$ and $\|f\| = \sup_x f(x)$. By ~~the~~ condition (i), ~~we have~~ $a < \infty$. Since $\mu(f) \leq a\|f\|$ for $\mu \in M$, it follows that

$$M \subseteq \prod_{f \in C^+(S)} [0, a\|f\|].$$

The right ~~jand~~ hand side is compact by Tychonov's theorem, ~~Hence~~ so M^* is also compact, ~~so~~ and $\bar{M}$ is compact by (6).

1.12. Topological vector spaces.

A topological vector space S over $\underline{\underline{C}}$ is defined to be a vector space over $\underline{\underline{C}}$ endowed with a Hausdorff topology under which the linear operation

$$\ell : C \times C \times S \times S \to S$$
$$(\alpha_1, \alpha_2, \xi_1, \xi_2) \mapsto \alpha_1 \xi_1 + \alpha_2 \xi_2$$

is continuous ; this ensures that vector addition and scalar multiplication are continuous. Similarly we define a topological vector space over $\underline{\underline{R}}$. In this section we discuss the properties of topological vector spaces over $\underline{\underline{C}}$; a similar discussion can be made for topological vector spaces over $\underline{\underline{R}}$.

Throughout this section we use the following notation :

S : a topological vector space over $\underline{\underline{C}}$,

$\xi, \eta, \ldots$: points of S,

$A, B, U, V, \ldots$: subsets of S,

$\alpha, \beta, \ldots$: complex numvers,

$A + \xi_0 = \{\xi + \xi_0 : \xi \in A\}$, (algebraic sum)

$\alpha A + \beta B = \{\alpha x + \beta y : x \in A, y \in B\}$

$\mathcal{U}(\xi)$ = hte neighborhoods of ξ.

Since the map $\xi \to \xi + \xi_0$ is bicontinuous, $U + \xi \in \mathcal{U}(\xi)$ if and only if $U \in \mathcal{U}(0)$. Similarly, if $\alpha \neq 0$, $\alpha U \in \mathcal{U}(0)$ if and only if $U \in \mathcal{U}(0)$

A general sequence $\{\xi_\alpha\}_{\alpha \in A}$, A being directed, is called <u>Canchy</u> if for every $U \in \mathcal{U}(0)$ there exists $\alpha_0 = \alpha_0(U)$ such that

$$\xi_\alpha - \xi_\beta \in U \quad \text{whenver} \quad \alpha, \beta \geq \alpha_0$$

If every general Cauchy sequence in S is convergent, S is called complete.

A subset A of S is called

balanced if

$$\alpha A \subset A \quad \text{whenever} \quad |\alpha| \leq 1,$$

convex if

$$\alpha A + (1-\alpha) A \subset A \quad \text{whenever} \quad 0 < \alpha < 1,$$

and bounded if for every $U \in \mathcal{U}(0)$ there exists an $\alpha \in (0, \infty)$ such that $A \subset \alpha U$.

S is called locally convex if the convex neighborhoods of 0 form a base of $\mathcal{U}(0)$. We now prove that

(1) If S is locally convex, the balanced convex neighborhoods of 0 form a base of $\mathcal{U}(0)$.

Since S is locally convex, it is enough to show that every convex neighborhood U of 0 contains a balanced convex neighborhood W of 0. Also the map $(\alpha, \xi) \longrightarrow \alpha\xi$ is continuous, so we can find $\varepsilon > 0$ and a neighborhood $V \in \mathcal{U}(0)$ such that

$$|\alpha| = \varepsilon \Rightarrow \alpha V \subset U,$$

and hence

(2) $|\beta| = 1 \Rightarrow \overset{\varepsilon}{\cancel{\alpha}} V \subset \beta U.$

Let W denote the interior of the intersection of all βU : $|\beta| = 1$. then $W \supset \varepsilon V$ by (2). Thus $W \in \mathcal{U}(0)$. It is easy to check that W is balanced and convex, which completes the proof of (1).

Let T be a vector space over $\mathbb{C}$, where no topology is given. A map $p : T \rightarrow [0, \infty)$ is called a semi-norm on T if the following conditions are satisfied :

(p.1) $\quad p(\alpha\xi) = |\alpha| p(\xi),$

(p.2) $\quad p(\xi + \eta) \leq p(\xi) + p(\eta).$

A family P of semi-norms on T is called sepatating, if

$p(x) = 0$ for every $p \in P \Rightarrow x = 0$.

For a given separating family P of semi-norms on T we may define a topology τ_P by the system of neighborhoods

$$U_{p_1, p_2, \ldots, p_n, \varepsilon}(\xi_0) = \{ \xi \in S : p_i(\xi - \xi_0) < \varepsilon, \ = 1, 2, \ldots, n \}$$

$$n = 1, 2, \ldots, \ p_i \in P, \quad \varepsilon > 0.$$

It is easy to see that the vector space T with the topology τ_P is a locally convex topological vector space Conversely we have

Theorem 1.12.1. Every locally convex topological vector space S carries a separating family P of semi-norms determining its topology.

Proof. We will sketch the proof. Let $\mathcal{W}$ be the family of all balanced convex neighborhoods of 0. By virtue of (1) $\mathcal{W}$ is a basis of $\mathcal{U}(0)$. For each $W \in \mathcal{W}$ set

$$P_W(\xi) = \inf \{ \alpha : \alpha W \ni \xi \}.$$

Then $P = \{P_W\}_{W \in \mathcal{W}}$ is a sepatating family of semi-norms on S and the topology τ_P determined by P coincides with the original topology in S.

A topological vector space is called an F-space if it is complete and metrizable. An F-space is called a Frechet space if it is locally convex. (This terminology follows Rudin [1] ; authors vary in their definition of F- and Frechet spaces.)

Let T and S be topological vector spaces. If $T \subset S$ (as sets). and if the linear operation and the topology on T are ~~incluced~~ induced from those on S, then T is called a <u>subspace</u> of S. Every subset T of S closed under the linear operation is a subspace of S when the linear operation and the topology on T are induced from those on S.

A topological vector space expressible as the union of an increasing sequence of Frechet subspaces is called an LF-space.

<u>Theorem 1.12.1</u>. Every LF-space is locally convex and complete.

<u>Theorem 1.12.2</u>. Let $\{S_n\}_{n = 1,2,\ldots}$ be a sequence of Frechet spaces such that S_n is a subspace of S_{n+1} for every n. Then the union $S = \cup_n S_n$ with the following linear operation and the topology is an LF-space :

(i) $\zeta = \alpha\xi + \beta\eta$ in $S \Leftrightarrow$ this holds in some S_n ,

(ii) G is open in $S \Leftrightarrow G \cap S_n$ is open in S_n for every n.

<u>Theorem 1.12.3</u>. Let S be an LF-space expressible as the union of an increasing sequence of Frechet subspaces $S_1, S_2, \ldots$. Then a linear functional $\ell : S \to \underline{C}$ is continuous if and only if the restirction $\ell|_{S_n} : S_n \to \underline{C}$ is continuous. We refer the reader to Treves [1] for the proof of these theorems.

Let S be a topological vector space. The set of all continuous linear functionals $x : S \to \mathbb{C}$ is called the dual space of S, denoted by S'. S' is a vector space with the usual linear operation

$$(\alpha_1 x_1 + \alpha_2 x_2)(\xi) = \alpha_1 x_1(\xi) + \alpha_2 x_2(\xi) \quad \text{for every } \xi \in S.$$

We may define many topologies on S' that make S' into a locally convex topological vector space. Among such topologies the following are most important.

(i) The <u>strong topology</u> on S' is defined by the family of semi-noms

$$p_B(x) = \sup_{\xi \in B} |x(\xi)|$$

where B rums over all bounded subsets of S. Note that every continuous linear functional is bounded on some $U \in \mathcal{U}(0)$ and therefore bounded on any bounded set B, so $p_B(x) < \infty$.

(ii) The <u>topology of uniform convergence on compacts</u> on S' is defined by the family of semi-norms

$$p_K(x) = \sup_{\xi \in K} |x(\xi)|$$

where K runs over all compact subsets of S.

(iii) The <u>weak* topology</u> on S' is defined by the family of semi-norms

$$p_\xi(x) = |x(\xi)| , \quad \xi \in S.$$

Among these three topologies (i) is the strongest and (iii) is the weakest.

1.13. The L^p spaces.

Let $T = (T, \mu)$ be a measure space, where μ is a finite measure. The set of all μ-measurable real (resp. complex) functions on T is denoted by

$$L^0 = L^0(T, \mu) \quad (\text{resp. } L^0_{\underline{\underline{C}}} = L^0_{\underline{\underline{C}}}(T, \mu)) \;;$$

as usual two μ-equivalent (i.e. equal a.e.(μ)) functions are identified. We will discuss L^0 in this section, but a similar discussion holds on $L^0_{\underline{\underline{C}}}$.

The set L^0 with the usual linear operation is a vector space over $\underline{\underline{R}}$. We will define a topology which makes L^0 into an F-space. Set

$$\|x\|_0 = \int_T [|x(t)| \wedge 1] \, \mu(dt).$$

$\| \;\|_0$ is not a norm on L^0 but it does have the following properties:

(i) $0 \leq \|x\|_0 \leq 1$, and $\|x\|_0 = 0 \Leftrightarrow x = 0$ (i.e. $x(t) = 0$ a.e.(μ)).

(ii) $\|\alpha x\|_0 = \|x\|_0$ if $|\alpha| = 1$,

(iii) $\|x + y\|_0 \leq \|x\|_0 + \|y\|_0$,

(iv) $\|\alpha x\|_0 \leq (|\alpha| + 1)\|x\|_0$,

(v) $\alpha_n \to 0 \Rightarrow \|\alpha_n x\|_0 \to 0$,

(vi) $\lim_{n,m\to\infty} \|x_n - x_m\|_0 = 0 \Rightarrow \lim_{n\to\infty} \|x_n - x\|_0 = 0$ for some $x \in L^0$

(i), (ii) and (iii) are obvious. (iv) follows from the obvious inequality $(a\, b) \wedge 1 \leq (a+1)(b \wedge 1)$ for $a, b \geq 0$.
(v) follows from the bounded convergence theorem for integrals.
To prove (vi) take a subsequence $y_k = x_{n_k}$, $k = 1, 2, \ldots,$ so that

$$\| y_{k+1} - y_k \| < 2^{-k}, \quad k = 1,2,\ldots$$

Then

$$\int_T \sum_{k=1} \left[|y_{k+1}(t) - y_k(t)| \wedge 1 \right] \mu(dt) = \sum_{k=1}^{\infty} \| y_{k+1} - y_k \|_0 < 1.$$

Hence

(1) $$\sum_{k=1} \left[|y_{k+1}(t) - y_k(t)| \wedge 1 \right] < \infty$$

holds a.e.(μ). If (1) holds, $|y_{k+1}(t) - y_k(t)| < 1$ for sufficiently large k , so

$$\sum_{k=1}^{\infty} |y_{k+1}(t) - y_k(t)| < \infty$$

holds a.e. (μ). This implies that

$$y_k = y_1 + (y_2 - y_1) + (y_3 - y_2) + \ldots + (y_k - y_{k-1}), \quad k = 1,2,\ldots$$

converges to some $x \in L^0$ a.e. (μ). Using the bounded convergence theorem, we can show that

$$\| x_n - x \|_0 \leq \lim_k \| x_n - y_k \|_0 ,$$

so

$$\overline{\lim_n} \| x_n - x \|_0 \leq \overline{\lim_n} \lim_k \| x_n - y_k \|_0 \leq \lim_{n,m \to \infty} \| x_n - x_m \| = 0,$$

which proves (v).

Defining

$$\rho_0(x,y) = \| x - y \|_0 ,$$

we obtain a metric on L^0 by (i), (ii) and (iii), so we may endow L^0 with the ρ_0-topology . Observing that

$$\varepsilon \mu\{t : |x(t)| \geq \varepsilon\} \leq \| x \|_0 \leq \varepsilon \mu(S) + \mu\{t : |x(t)| > \varepsilon\}, \quad \cancel{\varepsilon > 0,} \; 0 < \varepsilon \leq 1$$

we see that $x_n \to x$ in the ρ_0-topology if and only if

$$\lim_{n\to\infty} \mu\{t : |x_n(t)-x(t)| > \varepsilon\} = 0 \quad \text{for every} \quad \varepsilon>0.$$

In view of this fact we often call the ρ_0-topology the <u>topology of convergence in measure.</u>

<u>Theorem 1.13.1.</u> The vector space L^0 with the ρ_0-topology is an F-space.

Proof. The map $(x,y) \to x+y$ is continuous by (iii). Observing that

$$\|\alpha_n x_n - \alpha x\|_0 \leq \|\alpha_n(x_n-x)\|_0 + \|(\alpha_n-\alpha)\, x\|$$

and using (iv) and (v), we see that the map $(\alpha,x) \to \alpha x$ is also continuous. Therefore L^0 is a linear topological space. L^0 is evidently metrizable, and complete by (vi), so L^0 is an F-space.

The following example shows that $L^0(T,\mu)$ is not, in general, locally convex.

<u>Example.</u> $L^0(\underline{\underline{I}},\lambda)$ is not locally convex. (λ denotes Lebesgue measure).

<u>Proof.</u> Suppose that $L^0(\underline{\underline{I}},\lambda)$ is locally convex. Since $1 \neq 0$, both being regarded as members of $L^0(\underline{\underline{I}},\lambda)$, there must be a convex neighborhood V of 0 such that $1 \notin V$.

$$U := \left\{x : \|x\|_0 < \frac{1}{n}\right\} \subset V$$

and consider

$$x_i = (n+1)\, 1_{\left[\frac{i-1}{n+1},\, \frac{i}{n+1}\right)}, \quad i = 1,2,\ldots, n+1.$$

Then

(1) $\quad \|x_i\|_0 = \frac{1}{n+1}$ for every i

and

(2) $$1 = \frac{1}{n+1}\sum_{i=1}^{n+1} x_i .$$

From (1) we get $x_i \in U \subset V$, so (2), combined with convexity of V, implies that $1 \in V$, a contradiction.

For $x \in L^0$, set

$$\| x \|_p = \begin{cases} \left(\int_T |x(t)|^p \mu(dt) \right)^{1/p} & (1 \leq p < \infty) \\ \operatorname*{ess.\,sup}_{t \in T} x(t) = \inf \{ \alpha : |x(t)| \leq \alpha \text{ a.e.}(\mu) \} & (p = \infty) \end{cases}$$

and define $L^p = L^p(S, \mu)$ by

$$L^p = \{ x \in L^0 : \| x \|_p < \infty \}$$

It is well-known that the space L^p with the norm $\| \ \|_p$ is a Banach space.

2. Polish spaces, standard spaces and analytic spaces.

A Hausdorff topological space is called

Polish if it is homeomorphic to a complete separable metric space,

standard if it is 1-1 dominated by a complete separable metric space (i.e. it is 1-1 dominated by a Polish space), and

analytic if it is dominated by a complete separable metric space, (i.e. it is dominated by a Polish space).

It is obvious that

$$\{\text{Polish spaces}\} \subset \{\text{standard spaces}\} \subset \{\text{analytic spaces}\}.$$

These topological spaces have nice properties related to Borel structures and measures. The special topological spaces listed in § 1.4. are Polish except for the spaces $\underline{\underline{Q}}$, $\mathfrak{D}'(a)$ and $\mathfrak{D}'$ which are standard. Practically all topological spaces appearing in probability theory are standard.

A Borel space is called a standard (resp. analytic) Borel space, if it is Borel isomorphic with a standard (resp. analytic) space with the topological σ-algebra. It is obvious that

$$\{\text{standard Borel spaces}\} \subset \{\text{analytic Borel spaces}\}.$$

These Borel spaces also have some nice properties which can be derived from the properties of standard or analytic spaces.

2.1. Metric spaces.

A set S endowed with a metric ρ is called a metric space, denoted by $S(\rho)$ or (S,ρ). $S(\rho)$ is regarded as a Hausdorff topological space with the ρ-topology. If every ρ-Cauchy sequence in $S(\rho)$ converges to a point with respect to the ρ-topology, then ρ is called a complete metric and $S(\rho)$ is called a complete metric space.

Let $S(\rho)$ be a metric space. The ε-neighborhood of $a \in S$, the closed ε-neighborhood of $a \in S$ and the diameter of $A \subset S$ are denoted by $U(a, \varepsilon)$, $\bar{U}(a, \varepsilon)$ and $d(A)$ respectively:

$$U(a, \varepsilon) = \{x \in S : \rho(x,a) < \varepsilon\},$$

$$\bar{U}(a, \varepsilon) = \{x \in S : \rho(x,a) \leq \varepsilon\},$$

$$d(A) = \sup \{\rho(x,y) : x,y \in A\}.$$

We often include the suffix ρ to indicate the metric referred to; for example $U_\rho(a, \varepsilon)$. The distance between $a \in S$ and $B \subset S$ (or between $A \subset S$ and $B \subset S$) is denoted by $\rho(a, A)$ (or $\rho(A, B)$):

$$\rho(a,B) = \inf \{\rho(a,b) : b \in B\},$$

$$\rho(A,B) = \inf \{\rho(a,b) : a \in A, b \in B\}.$$

Theorem 2.1.1. Let $S(\rho)$ be a complete metric space.

(i) (The Cantor intersection theorem). If $\{F_n\}$ is a decreasing sequence of non-empty closed sets with $d(F_n) \to 0$, then the intersection $\bigcap_n F_n$ consists of a single point. Denote the point by a. Thus $F_n \downarrow a$.

(ii) (The Baire category theorem). If $\{F_n\}$ is a sequence of closed sets covering S, then at least one F_n includes a non-empty open set.

Proof.

(i) Take $a_n \in F_n$, $n=1,2,\ldots$. Then

$$a_n, a_m \in F_k \quad \text{for } n, m \geq k.$$

Therefore $\rho(a_n, a_m) \to 0$ $(n, m \to \infty)$ as $\rho(F_k) \to 0$. Since ρ is complete, we can find $a \in S$ such that $a_n \to a$. But $a_n \in F_k$ for $n \geq k$, so we have $a \in F_k$. Therefore $a \in \bigcap_k F_k$.

Since $\rho(F_k) \to 0$, $\{a\} = \bigcap_k F_k$ and $F_k \downarrow a$.

(ii) Suppose that none of the F_n, $n=1,2,\ldots,$ includes a non-empty open set. Take a point $x_1 \in S$ and a positive number $r_1 < 1$. Then $U(x_1,r_1) \setminus F_1$ is a non-empty open set. Take a point x_2 in this set and choose a positive number $r_2 < 1/2$ in such a way that

$$\bar{U}(x_2,r_2) \subset U(x_1,r_1) \setminus F_1 .$$

Then $U(x_2,r_2) \setminus F_2$ is a non-empty open set. Continuing this, we can find $x_n \in S$ and $r_n \in (0, 1/n)$, $n=1,2,\ldots,$ such that

$$\bar{U}(x_{n+1}, r_{n+1}) \subset U(x_n, \overset{r}{\cancel{F}}_n) \setminus F_n, \quad n=1,2,\ldots .$$

Applying the Cantor intersection theorem to $\{\bar{U}(x_n, r_n)\}_n$, we can find a point $x \in \bigcap_n \bar{U}(x_n,r_n)$. Then $x \notin \bigcup_n F_n$, contrary to the assumption.

Let S be a topological space and let ρ be a metric on S . If the ρ-topology on S is the same as the given topology on S , then ρ is called a <u>compatible metric</u> on S . A topological space S is called <u>metrizable</u> (resp. <u>completely metrizable</u>) if there exists a compatible metric (resp. a complete compatible metric) on S . It should be noted that the fact S is completely metrizable does not imply that every compatible metric on S is complete. For example, consider the <u>positive half-line</u> $\underline{\underline{R}}^+ = (0,\infty)$. This is completely metrizable, because $\rho(x,y) := |\log x - \log y|$ is a complete compatible metric on $\underline{\underline{R}}^+$. But the usual metric $\rho_0(x,y) := |x-y|$ is compatible but not complete, because $\{1/n\}_{n=1,2,\ldots}$ is ρ_0-Cauchy but does not converge to any point in $\underline{\underline{R}}^+$.

If ρ is a compatible metric on S, then $\rho \wedge 1$ is also a compatible metric on S. Therefore every metrizable space has a compatible metric bounded by 1, and similarly for a complete compatible metric.

2.2. Polish spaces.

Let S be a Polish space. Then S is homeomorphic to a complete separable metric space $S'(\rho')$ under a bicontinuous map $f : S \to S'$. Thus

$$\rho(x,y) := \rho'(f(x),f(y))$$

defines a complete compatible metric on S. Since $S'(\rho')$ is separable, $S(\rho)$ is also separable and hence S has a countable open base. Therefore a Polish space is a completely metrizable space with a countable open base.

Let S be a completely metrizable space with a countable open base. Then S has a complete compatible metric ρ and $S(\rho)$ is complete and separable. Since the identity map $i : S \to S(\rho)$ is bicontinuous by compatibility of ρ, S is Polish.

By the above observation we can define a Polish space to be a completely metrizable space with a countable open base.

From the definition we see that every Polish space has all the topological properties of a complete separable metric space.
For example:

(i) Every Polish space is normal and fully Lindelöf,

(ii) On a Polish space every closed set is G_δ,

(iii) The Baire category theorem (Theorem 2.1.1.(ii)) holds for a Polish space,

Theorem 2.2.1. Every closed subset T of a Polish space S is Polish, where T is endowed with the relative topology.

Proof. Let ρ be a complete compatible metric on S. Then the restriction ρ_T of ρ to T is a compatible metric on T. Since T is closed in S, it is easy to see that ρ_T is complete. T has a countable open base as a subspace of S. Therefore T is Polish.

Theorem 2.2.2. Every countable disjoint sum of Polish spaces is Polish.

Proof. Let S_n, $n=1,2,\ldots$, be Polish spaces and let ρ_n be a complete compatible metric on S_n, bounded by 1, for each n. Let S be the disjoint sum of S_n, $n=1,2,\ldots$. Then a point in S is of the form (x, n) where $n=1,2,\ldots$ and $x \in S_n$; see § 1.6. Define a metric ρ on S by

$$\begin{aligned}\rho((x,n),(y,m)) &= \rho_n(x,y) && \text{if } m=n\\ &= 1 && \text{if } m\neq n\,.\end{aligned}$$

It is easy to check that ρ is a complete compatible metric on S and that S has a countable open base. Therefore S is Polish.

Theorem 2.2.3. Every countable product of Polish spaces is Polish, where the product space is endowed with the product topology.

Proof. Let S_n and ρ_n be as in the proof of the above theorem. Then

$$\rho((x_n),(y_n)) := \sum_n 2^{-n} \rho_n(x_n,y_n), \quad x_n,y_n \in S_n,\ n=1,2,\ldots,$$

defines a complete compatible metric on $S = \prod_n S_n$. It is easy to see that S has a countable open base.

Theorem 2.2.4. Every countable projective limit of Polish spaces is Polish. ~~[illegible]~~ cf. (1.5), (1.6)

Proof. This follows at once from Theorems 2.2.3. and 2.2.1., because the projective limit of Hausdorff topological spaces is a closed subset of their product space.

Theorem 2.2.5. Every compact metrizable space is Polish.

Remark. By Urysohn's metrization theorem it is obvious that a compact Hausdorff space is metrizable if and only if it has a countable open base.

Proof. Let S be a compact metrizable space, ρ a compatible metric on S, and let x_n be a ρ-Cauchy sequence. Since S is compact, $\{x_n\}$ has a subsequence converging to a point $x \in S$. It is easy to see that $x_n \to x$. Therefore ρ is complete. By the above remark S has a countable open base.

Now we will examine which of the special spaces listed in § 1.4. are Polish.

(i) $\underline{\underline{R}}$, $\overline{\overline{\underline{\underline{R}}}}$ and $\underline{\underline{C}}$ are Polish.

(ii) $\underline{\underline{I}}$, $\underline{\underline{N}}$, $\underline{\underline{Z}}$, $\underline{\underline{2}}$ and $\underline{\underline{K}}$ are Polish by Theorem 2.2.1., because they are closed in $\underline{\underline{R}}$.

(iii) $\underline{\underline{R}}^n$, $\underline{\underline{C}}^n$, $\underline{\underline{2}}^n$ and $\underline{\underline{N}}^n$ $(n=1,2,\dots,\infty)$ are Polish by Theorem 2.2.3.

(iv) $\underline{\underline{J}}$ is Polish, because it is homeomorphic to $\underline{\underline{N}}^\infty$.

(v) $\underline{\underline{Q}}$ is not Polish, because the Baire category theorem ([illegible]) does not hold on $\underline{\underline{Q}}$; consider the covering of $\underline{\underline{Q}}$ by all singletons.

In Chapter 2 we will prove that the spaces $C[0,1]$, $D[0,1]$ and $L^p[0,1]$ $(1 \le p < \infty)$ are Polish.

2.3. Polish subsets

Let S be a topological space. A subset T of S is called Polish, if the set T with the relative topology is a Polish space.

Theorem 2.3.1. (Alexandrov). A subset T of a Polish space S is Polish if and only if T is G_δ in S.

Proof. Since T always has a countable open base, being a subspace of S, the Polish property of T follows from the existence of a complete compatible metric on T.

First we will prove the following:

(1) Every open subset T of S is Polish.

Let ρ be a complete compatible metric on S and let ρ_T be the restriction of ρ to T. ρ_T is compatible with the relative topology in T but is not complete except in the trivial case $T = S$. We will modify ρ_T to construct a complete compatible metric ρ'_T on T. First define

$$f(x) := \rho(x, S-T) = \inf\{\rho(x,y) : y \in S-T\}.$$

Then $f(x)$ is continuous. $f(x) > 0$ if and only if $x \in T$, because $S - T$ is closed. Therefore $g(x) := 1/f(x)$ $(x \in T)$ is continuous on T. Define a new metric ρ'_T on T by

$$\rho'_T(x,y) = \rho_T(x,y) + |g(x) - g(y)| .$$

If $\rho_T(x_n,x) \to 0$, then $\rho'_T(x_n,x) \to 0$ by the continuity of g, and the converse is obvious as $\rho_T \le \rho'_T$. It follows that ρ'_T is compatible. We now prove that ρ'_T is complete.

Let $\{x_n\}$ be a ρ_T'-Cauchy sequence in T. As $\rho_T \leq \rho_T'$ $\{x_n\}$ is ρ_T-Cauchy in T, i.e. ρ-Cauchy in S. Therefore $\{x_n\}$ converges to a point $x \in S$. If $x \in S-T$, then

$$f(x_n) \to f(x) = 0 , \quad \text{i.e.} \quad g(x_n) \to \infty ,$$

in contradiction to

$$|g(x_n) - g(x_m)| \leq \rho_T'(x_n,x_m) \to 0 \quad (n,m \to \infty) .$$

Therefore x must be in T, which shows that ρ_T' is complete. This completes the proof of (1).

Second we will show the following:

(2) If $T_n \subset S$, $n=1,2,\ldots$, are Polish, then the intersection $T := \bigcap_n T_n$ is Polish.

Let D be the diagonal set of the product space $\Pi := \prod_n T_n$. Then D is closed in Π and homeomorphic to T by Theorem 1.5.2. Since D is Polish by Theorem 2.2.3. and 2.2.1, T is also Polish, as desired.

By (1) and (2), every G_δ subset of S is Polish. To complete the proof of our theorem, it is enough to show that if T is Polish, then T is G_δ in S. Take a complete compatible metric ρ_T on T and denote the ρ_T-diameter of $A \subset T$ by $d_T(A)$. Let $\mathcal{U}_n$ denote the class of all sets U open in S such that

$$d_T(U \cap T) < \frac{1}{n} ,$$

and let G_n be the union of all $U \in \mathcal{U}_n$. Then G_n is open in S, and the closure $\overline{T}$ of T in S is G_δ in S, being a closed subset of S. To prove that T is G_δ in S, it is enough to show that

(3) $T = \overline{T} \cap (\bigcap_n G_n)$.

Suppose that $x \in T$, and take a neighborhood $V_n = V_n(x)$ in T such that $d_T(V_n) < 1/n$. Then $V_n = U_n \cap T$ for some neighborhood $U_n = U_n(x)$ in S . Thus $d_T(U_n \cap T) < 1/n$. This shows that $U_n \in \mathcal{U}_n$, whence $x \in G_n$. Since $x \in T \subset \overline{T}$, x belongs to the right hand side of (3), call it R,

Suppose conversely that $x \in R$. Then for every n , we can find a neighborhood $U_n = U_n(x)$ in S such that

$$d_T(U_n \cap T) < 1/n .$$

This inequality continues to hold, even if we replace U_n by a smaller ? neighborhood of x for each n . Therefore we can assume that $U_n \downarrow x$. Since $x \in \overline{T}$, $U_n \cap T$ contains at least one point, say x_n . As $\{U_n \cap T\}_n$ is decreasing,

$$x_m, x_n \in U_k \cap T \qquad \text{for } m, n > k$$

and hence

$$\rho_T(x_m, x_n) < \frac{1}{k} \qquad \text{for } m, n > k .$$

Since ρ_T is complete, we have $y \in T$ such that $x_n \to y$ in T . Therefore $x_n \to y$ in S . But $x_n \to x$ since $U_n \to x$. We conclude that $x = y \in T$, completing the proof of (3), and the theorem..

<u>Theorem 2.3.2. (Alexandroff-Urysohn).</u> A topological space is Polish, if and only if it is homeomorphic to a G_δ subset of $\underline{I}^\infty$.

Proof. Since $\underline{\underline{I}} = [0,1]$ is Polish, every G_δ subset of $\underline{\underline{I}}^\infty$ is Polish by Theorems 2.2.3 and 2.3.1, and the "if" part of the theorem follows immediately. To establish the other half of the theorem, let S be a Polish space and ρ a complete compatible metric on S bounded by 1. Take a sequence $\{a_n\}$ dense in S and define a map

$$f : S \to I^\infty$$

$$x \mapsto (\rho(x,a_1), \rho(x,a_2), \ldots) .$$

Let B denote the image $f(S)$ and g the restriction $f|_{S,B}$. By a routine argument we have that the map $g : S \to B$ is bicontinuous. Therefore S is homeomorphic to B. Since S is Polish, B is also Polish. Hence B is G_δ in $\underline{\underline{I}}^\infty$ by Theorem 2.3.1, completing the proof.

A topological space is called <u>σ-compact</u>, if it is expressible as a countable union of compact subsets.

<u>Theorem 2.3.3.</u> <u>Every locally compact, σ-compact metrizable space is Polish.</u>

<u>Proof</u>. Such a space S is open in its one-point compactification $\overline{S} = S \cup \{\infty\}$ and $\overline{S}$ is metrizable. Since $\overline{S}$ is Polish by Theorem 2.2.5, S is Polish by Theorem 2.3.1.

2.4. 0-dimensional Polish spaces.

A topological space is called 0- dimensional if there is an open base consisting of simultaneously open and closed sets. This property is inherited by subspaces and product spaces. Therefore $\underline{\underline{N}}^{\omega}$, $\underline{\underline{2}}^{\infty}$ and their G_δ subsets are 0-dimensional Polish spaces.

Theorem 2.4.1. Every Polish space is 1-1 dominated by a 0-dimenensional Polish space.

1-1

Proof. Let P_0 denote the property of being dominated by a 0-dimensional Polish space. It is easy to see that P_0 is inherited by countable products and G_δ subdets. Since every Polish space is homomorphic to a G_δ subsets of $\underline{\underline{I}}^{\infty}$, it remains only to prove that $\underline{\underline{I}} \equiv [0,1]$ has property P_0. Consider the map

$$f \;:\; \underline{\underline{2}}^{\infty} \longrightarrow \underline{\underline{I}}$$
$$(i_n) \mapsto \sum_{n=1}^{\infty} 2^{-n}\, i_n \,.$$

This is continous and surjective but not bijective. Let A be the set of all points in $\underline{\underline{I}}$ expressible as $k/2^n$ ($k = 1,2,\ldots, 2^n - 1$; $n = 1,2,\ldots$). Then A is countable and

$f^{-1}(a)$ consists of two points for $a \in A$,

and $f^{-1}(b)$ consists of one point for $b \in I - A$.

Choose a point $\xi(a)$ in $f^{-1}(a)$ for each a in A and Let

$$A' = \{\xi(a) : a \in A\} \quad \text{and} \quad S_0 = \underline{\underline{2}}^{\infty} - A'$$

Then the restriction $f_0 := f|_{S_0} : S_0 \longrightarrow \underline{\underline{I}}$ is a continuous bijection. Since A' is countable, S_0 is G_δ in $\underline{\underline{2}}^{\infty}$ and hence is Polish. This proves that $\underline{\underline{I}}$ has property P_0, completing the proof of the theorem.

Theorem 2.4.2. Every Polish space S is 1-1 dominated by a closed subset of $\underline{N}^{\infty}$.

Proof. By the previous theorem we can assume that S is 0-dimensional. Let ρ be a complete compatible metric on S and let d denote the ρ-diameter. For every $\varepsilon > 0$, we can find a sequence of simultaneously open and closed sets $U_n(\varepsilon)$, $n=1,2,\ldots$, with $d(U_n(\varepsilon)) < \varepsilon$ such that

$$S = \bigcup_n U_n(\varepsilon) ,$$

because S is 0-dimensional and fully Lindelöf. We can assume that $\{U_n(\varepsilon)\}_n$ is disjoint for each $\varepsilon > 0$, by replacing $U_n(\varepsilon)$ by $U_n(\varepsilon) - \bigcup_{k<n} U_k(\varepsilon)$ if necessary. Then the Souslin scheme

$$A_{n_1n_2\ldots n_k} = U_{n_1}(1) \cap U_{n_2}(\tfrac{1}{2}) \cap \ldots \cap U_{n_k}(\tfrac{1}{k})$$

is decreasing and disjoint. Since every $x \in S$ belongs to some $U_n(\varepsilon)$ for every $\varepsilon > 0$, we have

$$(1) \qquad S = \sum_{\underline{n} \in \underline{N}^{\infty}} \bigcap_k A_{n_1n_2\ldots n_k} , \text{ where } \underline{n}=(n_1,n_2,\ldots) .$$

Let F denote the set of all $\underline{n} \in \underline{N}^{\infty}$ such that $A_{n_1n_2\ldots n_k} \neq \emptyset$ for every k . Since $\{A_{n_1n_2\ldots n_k}\}$ is a decreasing sequence of non-empty closed sets such that $d(A_{n_1n_2\ldots n_k}) < k^{-1}$, we can use the Cantor intersection theorem (Theorem 2.1.1.(i)) to conclude that for every $\underline{n} \in F$, $\bigcap_k A_{n_1n_2\ldots n_k}$ consists of exactly one point, which we denote by $f(\underline{n})$. Then the map $f\colon F \to S$ is bijective by (1) . Since for every $\underline{n} = (n_k) \in F$ the sets

$$F_{n_1n_2\ldots n_k} = \underline{N}_{n_1n_2\ldots n_k} \cap F , \ k=1,2,\ldots ,$$

are neighborhoods of $\underline{\underline{n}}$ in F and since

$$f(F_{n_1 n_2 \ldots n_k}) \subset A_{n_1 n_2 \ldots n_k} \quad \text{and} \quad d(A_{n_1 n_2 \ldots n_k}) < k^{-1} ,$$

the map $f : F \to S$ is continuous. Therefore S is dominated by F. It remains only to prove that F is closed in $\underline{\underline{N}}^{\infty}$, or equivalently F^c is open in $\underline{\underline{N}}^{\infty}$. Let $\underline{\underline{n}} = (n_k) \in F^c$. Then $A_{n_1 n_2 \ldots n_j}$ is empty for some j. This implies that

$$(m_k) \in F^c \quad \text{if} \quad m_k = n_k \ , \ k=1,2,\ldots,j \ .$$

Therefore $N_{n_1 n_2 \ldots n_j} \subset F^c$, which proves that F^c is open.

<u>Theorem 2.4.3.</u> Every Polish space S is dominated by $\underline{\underline{N}}^{\infty}$.

<u>Proof</u>. Let ρ be a complete compatible metric on S. We construct a Souslin scheme $\mathcal{S} = \{A_{n_1 n_2 \ldots n_k}\}$ as follows. Take a sequence $\{a_n\}$ dense in S and let

$$A_n = \bar{U}(a_n, 1) \ , \quad n=1,2,\ldots \ .$$

If $A_{n_1 n_2 \ldots n_k}$ is constructed, take a sequence $\{b_n\}$ dense in this set and let

$$A_{n_1 n_2 \ldots n_k n} = A_{n_1 n_2 \ldots n_k} \cap \bar{U}\left(b_n, \frac{1}{k+1}\right), \ n=1,2,\ldots \ .$$

For every $\underline{\underline{n}} = (n_k) \in \underline{\underline{N}}$, $\{A_{n_1 n_2 \ldots n_k}\}_k$ is a decreasing sequence of non-empty closed sets with $d(A_{n_1 n_2 \ldots n_k}) \to 0 \ (k \to \infty)$. Therefore we can use the Cantor intersection theorem to conclude that for every $\underline{\underline{n}} \in \underline{\underline{N}}^{\infty}$, $\bigcap_k A_{n_1 n_2 \ldots n_k}$ consists of exactly one point, which we denote by $f(\underline{\underline{n}})$. Then $f : \underline{\underline{N}} \to S$ is a continuous surjection and therefore S is dominated by $\underline{\underline{N}}^{\infty}$.

2.5. Standard spaces and analytic spaces.

Since the relations $\prec$ and $\underset{1-1}{\prec}$ are transitive, we can derive the following facts at once from the definitions given at the beginning of this chapter.

(S.1) A Hausdorff space 1-1 dominated by a standard space is standard.

(S.2) A Hausdorff space 1-1 dominated by a closed subset of $\underline{\underline{N}}^{\infty}$ is standard.

(A.1) A Hausdorff space dominated by an analytic space is analytic.

(A.2) A Hausdorff space dominated by $\underline{\underline{N}}^{\infty}$ is analytic.

Let $\mathcal{O}$ and $\mathcal{O}'$ be topologies on a set S and suppose that $\mathcal{O}'$ is weaker than $\mathcal{O}$ (i.e. $\mathcal{O}' \subset \mathcal{O}$). Then the identity map $i : S(\mathcal{O}) \to S(\mathcal{O}')$ is a continuous bijection. Therefore we obtain the following facts.

(S.3) If S is standard, then the set S with a weaker Hausdorff topology is also standard.

(A.3) If S is analytic, then the set S with a weaker Hausdorff topology is also analytic.

As already mentioned, every Polish space is standard and every standard space is analytic.

A trivial example of a standard but non-Polish space is $\underline{\underline{Q}}$. $\underline{\underline{Q}}$ is standard because it is dominated by $\underline{\underline{N}}$, but not Polish (see the end of § 2.2.). A more interesting example is the space $C = C[0,1]$ with the topology of point-wise convergence, denoted by $\mathcal{O}_p$. In this topology the following sets form a neighborhood base of $x \in C$.

$$U_{t_1,t_2,\dots,t_n,\varepsilon}(x) = \{y \in C : |y(t_i) - x(t_i)| < \varepsilon,\ i=1,2,\dots,n\},$$

$$\varepsilon > 0\ ;\ n=1,2,\dots\ ;\ t_k \in [0,1]\ .$$

The topological space $C(\mathcal{O}_p)$ is Hausdorff but not metrizable, because there is no countable neighborhood base of any point $x \in C$. Therefore $C(\mathcal{O}_p)$ is not Polish. But the space C with the maximum norm topology $\mathcal{O}_m$ is Polish, because the set C with the maximum norm metric is a complete separable metric space. (See § 2.11). It is obvious that $\mathcal{O}_p$ is weaker than $\mathcal{O}_m$. Therefore $C(\mathcal{O}_p)$ is standard by (S.3)...

Examples of analytic but non-standard spaces and non-analytic spaces will be given in § 2.6.

We now present some general properties of analytic spaces and standard spaces.

<u>Theorem 2.5.1.</u> Every analytic space (and therefore every standard space) is fully Lindelöf.

<u>Proof.</u> Let S be analytic. Then we have a Polish space P and a continuous surjection $f : P \to S$. Let $\{G_i\}_{i \in I}$ be an arbitrary class of open subsets of S. Then $f^{-1}(G_i)$ is open in S for every i. Since P is fully Lindelöf, we can find a countable subset J of I such that

$$\bigcup_{i \in J} f^{-1}(G_i) = \bigcup_{i \in I} f^{-1}(G_i)\ .$$

Since f is surjective, this implies that

$$\bigcup_{i \in J} G_i = \bigcup_{i \in I} G_i\ ,$$

proving that S is fully Lindelöf.

Theorem 2.5.2. Every closed (or open or G_δ) subset of an analytic space is analytic and every closed (or open or G_δ) subset of a standard space is standard.

Note. Recall that a subset of a topological space is endowed with the relative topology.

Proof. Let S be analytic. Take a Polish space P and a continuous surjection $f : P \to S$. Let B be closed (or open or G_δ) in S. Then $A = f^{-1}(B)$ is also closed (or open or G_δ) in P. By Theorem 2.3.2. A is Polish. Since the restriction

$$f_{A,B} : A \twoheadrightarrow B$$

is continuous and surjective, B is analytic. The proof for standard spaces is analogous.

Theorem 2.5.3. Every countable disjoint sum of analytic spaces is analytic and every countable disjoint sum of standard space is standard.

Proof. Let S_n, $n=1,2,\ldots$, be analytic. For each n, take a Polish space P_n and a continuous surjection $f_n : P_n \to S_n$. Let P and S be the disjoint sums of $\{P_n\}$ and $\{S_n\}$ respectively. Then P is Polish and

$$f : P \longrightarrow S$$
$$(x,n) \longmapsto (f_n(x),n) \quad (x \quad P_n, n=1,2,\ldots)$$

is a continuous surjection. Therefore S is analytic. The proof for standard spaces is the same.

Theorem 2.5.4. Every countable product of analytic spaces is analytic and every countable product of standard spaces is standard.

Proof. Use the bilateral product map.

Theorem 2.5.5. Every countable projective limit of analytic spaces is analytic and every countable projective limit of standard spaces is standard.

Proof. Since every projective limit of Hausdorff spaces is a closed subset of their product, this theorems follows from Theorems 2.5.4. and 2.5.2. ?

Theorem 2.5.6. Every analytic space S with $\#S > \aleph_0$ includes a compact subset homeomorphic to $\underline{\underline{2}}^{\infty}$.

Proof. We have a complete separable metric space $P(\rho)$ and a continuous surjection $f : P \longrightarrow S$. For each y in S take a point $g(y)$ in $f^{-1}(y)$. Then the set $A = \{ g(y) : y \in S \} \subset P$ has the same cardinal number as S , i.e. A is not countable. Let C be the set of all points $x \in A$ such that at least one neighborhood $V(x)$ has at most countably many points in common with A . Pick such a neighborhood $V(x)$ for each $x \in C$. Then C is covered by $\{V(x), x \in C\}$. Since P is fully Lindelöf, we can find a sequence $\{V(x_n), n=1,2,\ldots\}$ covering C . Therefore

$$C \subset \bigcup_n V(x_n) \cap A .$$

This shows that C is countable. Let $B = A - C$. B has the following properties:

(B.1) for every neighborhood $U(x)$ of $x \in B$, $U(x) \cap B$ contains at least two distinct points (in fact, uncountably many points).

(B.2) for any distinct points $x, y \in B$, $f(x) \neq f(y)$.

Now we determine a diadic system of points and neighborhoods ?

$$x_{i_1 i_2 \ldots i_n} \in B, \quad U_{i_1 i_2 \ldots i_n} = U(x_{i_1 i_2 \ldots i_n}, r_n)$$

$$n = 1,2,\ldots ; \quad i_k = 0,1,$$

by induction. Pick two distinct points x_0 and x_1 in B. Then $f(x_0) \neq f(x_1)$. By the continuity of the map f we can find neighborhoods

$$U_0 = U(x_0, r_1), \quad U_1 = U(x_1, r_1) \quad (0 < r_1 < \tfrac{1}{2})$$

such that

$$f(\bar{U}_0) \cap f(\bar{U}_1) = \emptyset .$$

If $x_{i_1 i_2 \ldots i_n} \in B$ and $U_{i_1 i_2 \ldots i_n} = U(x_{i_1 i_2 \ldots i_n}, r_n)$ are determined, then we determine $x_{i_1 i_2 \ldots i_n i} \in B$ and $U_{i_1 i_2 \ldots i_n i}$ $(i=0,1)$ as follows. By property (B.2) we can pick two distinct points $x_{i_1 i_2 \ldots i_n i}$, $i=0,1$ in $U_{i_1 i_2 \ldots i_n} \cap B$. Then

$$f(x_{i_1 i_2 \ldots i_n 0}) \neq f(x_{i_1 i_2 \ldots i_n 1}) .$$

Therefore we can find neighborhoods

$$U_{i_1 i_2 \ldots i_n i} = U(x_{i_1 i_2 \ldots i_n i}, r_{n+1}) \quad (i=0,1 ; \; 0 < r_{n+1} < 2^{-n-1})$$

such that

$$\bar{U}_{i_1 i_2 \ldots i_n 0} \cup \bar{U}_{i_1 i_2 \ldots i_n 1} \subset U_{i_1 i_2 \ldots i_n},$$

and

$$f(\bar{U}_{i_1 i_2 \ldots i_n 0}) \cap f(\bar{U}_{i_1 i_2 \ldots i_n 1}) = \emptyset .$$

Using the Cantor intersection theorem, we can check that for each $\underline{\underline{i}} = (i_n)$ the intersection $\bigcap_n \bar{U}_{i_1 i_2 \ldots i_n}$ consists of exactly one point, which we denote by $h(\underline{\underline{i}})$. Since

$$h(\underline{\underline{2}}_{i_1 i_2 \ldots i_n}) \subset \bar{U}_{i_1 i_2 \ldots i_n} \quad \text{(see § 1.4. for } \underline{\underline{2}}_{i_1 i_2 \ldots i_n}) ,$$

the map $h : \underline{\underline{2}}^{\infty} \to P$ is continuous. Therefore $f \circ h : \underline{\underline{2}}^{\infty} \to S$ is continuous. Also, we have

$$(f \circ h)\,(\underline{\underline{2}}_{i_1 i_2 \ldots i_n}) \subset f(\bar{U}_{i_1 i_2 \ldots i_n}) ,$$

which implies that $f \circ h$ is injective. Therefore $K := (f \circ h)(\underline{\underline{2}}^{\infty})$ is a compact subset of S homeomorphic to $\underline{\underline{2}}^{\infty}$.

Theorem 2.5.7. Every compact analytic space is metrizable (and hence Polish by Theorem 2.2.5.).

Proof. Let S be a compact analytic space. Then S is normal. To prove that S is metrizable it is enough to show that S has a countable open base, because every normal topological space with a countable open base is metrizable by the Urysohn metrization theorem. The space $S^2 := S \times S$ is also a compact analytic space by Theorem 2.5.4. Therefore S^2 is compact and fully Lindelöf. The diagonal set D of S^2 is closed in S^2 and therefore D is also compact. The set $G = S^2 - D$ is open in S^2 . Since S^2 is compact and hence normal, for every $\xi \in G$, we can find a neighborhood $U(\xi)$ such that

$$\overline{U(\xi)} \subset G .$$

Since G is covered by $\{U(\xi), \xi \in G\}$ and since S^2 is fully Lindelöf, we have a sequence $\{\xi_n\} \subset G$ such that $\{U(\xi_n)\}_n$ covers G. Let G_n denote $\bigcup_{k=1}^n U(\xi_k)$. Then

$$\bar{G}_n = \bigcup_{k=1}^n \overline{U(\xi_k)} \subset G ,$$

i.e. $$D = S^2 - G \subset S^2 - \bar{G}_n .$$

But D is compact and $S^2 - \bar{G}_n$ is open. Therefore, for each n we can find a finite number of open sets $\{V_{ni}\}_i$ in S such that

$$D \subset \bigcup_i V_{ni}^2 \subset S^2 - \bar{G}_n \quad (V_{ij}^2 := V_{ij} \times V_{ij}) .$$

We will prove that the countable class $\{V_{ni}\}_{n,i}$ is an open base in S. Let x be any point in S and V any neighborhood of x in S. Then

$$\{x\} \times V^c \subset S^2 - D = G \quad \text{and} \quad G_n \uparrow G .$$

Since $\{x\} \times V^c$ is closed in S^2 and hence compact, we have

$$\{x\} \times V^c \subset G_m \quad \text{for some } m .$$

For this m, we take j so that $(x, x) \in V_{mj}^2$. Then

$$V_{mj}^2 \cap (\{x\} \times V^c) \subset (S^2 - \bar{G}_m) \cap G_m = \emptyset ,$$

i.e. $$\{x\} \times (V_{mj} \cap V^c) = \emptyset ,$$

i.e. $$V_{mj} \cap V^c = \emptyset .$$

This implies that $x \in V_{mj} \subset V$, proving that $\{V_{ni}\}_{n,i}$ is an open base.

<u>Theorem 2.5.8</u>. If S_n, $n=1,2,\ldots$, are analytic, then

$$\mathcal{B}(\prod_n S_n) = \prod_n \mathcal{B}(S_n) .$$

<u>Proof</u>. $\prod_n S_n$ is analytic by Theorem 2.5.4. and therefore it is fully Lindelöf by Theorem 2.5.1. We now use Theorem 1.4.4.(ii).